§9.4 Positive Series: Other Tests

Thm Ordinary Comparison Test (important)

Suppose that $0 \le a_n \le b_n$ for $n \ge N$ for some (large) $N \in \mathbb{N}$.

1) If $\sum b_n$ converges then $\sum a_n$ converges.
2) If $\sum a_n$ diverges then $\sum b_n$ diverges.

Pf

1) Recall the Bounded Sum Test from 9.3. It says that a series of nonnegative terms converges if and only if its partial sums are bounded above. Since $\sum b_n$ converges and $0 \le a_n \le b_n$ for $n \ge N$, we see that the Bounded Sum Test applies and implies the convergence of $\sum a_n$.

2) This statement is simply the contrapositive of (1). $\square$

Ex Does the series $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$ converge or diverge?

Solution The n^{th} term of the series is given by $\frac{1}{\sqrt{n}} = b_n$.

We can compare with the harmonic series $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ whose n^{th} term is $\frac{1}{n} = a_n$. Since $a_n = \frac{1}{n} \le \frac{1}{\sqrt{n}} = b_n$, and since the harmonic series diverges, part (2) of the comparison test $\Rightarrow \sum \frac{1}{\sqrt{n}}$ diverges.

Ex Does the series $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \frac{1}{37} + \cdots$ converge?

Solution

This series is $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$. Compare this with the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $p=2$, which converges. We see $\frac{1}{n^2+1} \le \frac{1}{n^2}$ for all n, thus part (1) of the comparison test $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

Thm Limit Comparison Test (important)

Suppose that $a_n \geq 0$, $b_n \geq 0$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

1) If $0 < L < \infty$, then Σa_n and Σb_n converge or diverge together.
2) If $L = 0$ and Σb_n converges then Σa_n converges.
3) If $L = \infty$ and Σb_n diverges then Σa_n diverges.

Ex. Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{3n+1}{n^3-4}$

Solution

Basically you want to compare with the simplest expression that has the same behavior as $n \to \infty$. So we just need to keep the highest degree terms in both the numerator and denominator.

Thus, $a_n = \frac{3n+1}{n^3-4} \sim \frac{n}{n^3} = \frac{1}{n^2} = b_n$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n+1}{n^3-4} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{3n^3+n^2}{n^3-4} = 3 \quad (0<3<\infty)$$

Since $\Sigma \frac{1}{n^2}$ is just the p-series with $p=2$, it converges, thus by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{3n+1}{n^3-4}$ $\boxed{\text{converges}}$.

Ex $\sum_{n=1}^{\infty} \frac{\sqrt{2n+1}}{n^2}$ $\quad b_n = \frac{n^{1/2}}{n^2} = \frac{1}{n^{3/2}}$ $\quad \lim_{n\to\infty} \frac{\sqrt{2n+1}}{n^2} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{\sqrt{2n^4+n^3}}{n^2} = \sqrt{2}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{2n+1}}{n^2}$ converges by LCT, since $\Sigma \frac{1}{n^{3/2}}$ is a p-series with $p = 3/2$.

Thm Ratio Test

Let $\sum a_n$ be a series of positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho \qquad \text{"rho"}$$

1) If $\rho < 1$ then series converges
2) If $\rho > 1$ then " diverges
3) If $\rho = 1$, the test is inconclusive.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

Solution

$$a_{n+1} = \frac{2^{n+1}}{(n+1)!} = \frac{2\cdot 2^n}{(n+1)\,n!}$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0 = \rho$$

$\rho < 1 \Rightarrow$ Ratio Test implies $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ $\boxed{\text{converges}}$.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^{20}}$

Solution

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{\cancel{n+1}}}{(n+1)^{20}}\cdot\frac{n^{20}}{\cancel{2^n}} = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^{20} = 2 = \rho > 1$$

$\boxed{\text{Thus } \displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^{20}} \text{ diverges by the ratio test}}$