# §9.3 Positive Series: The Integral Test

**Thm** Bounded Sum Test

"if and only if"

A series $\sum a_k$ of nonnegative terms converges $\Longleftrightarrow$ its accompanying sequence of partial sums is bounded above.

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**Example**

Show that the series $\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$ converges.

**Solution** Note that $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n$ "n factorial"

Also note that:

$$\sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2 \cdot 1} + \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} + \cdots$$

$$\begin{array}{ccccc} \frac{1}{1} & \frac{1}{2\cdot 1} & \frac{1}{3\cdot 2\cdot 1} & \frac{1}{4\cdot 3\cdot 2\cdot 1} \\ \parallel & \parallel & \wedge & \wedge \\ \frac{1}{2^0} & \frac{1}{2^1} & \frac{1}{2^2} & \frac{1}{2^3} \end{array}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 2$$

The bottom series is the geometric series with $a=1$, $r=\frac{1}{2}$ and thus sums to $\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$, therefore by the bounded sum test given above, the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

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**Thm** Integral Test (very important test)

Let $f$ be a continuous, positive, nonincreasing function on $[1, \infty)$ and suppose $a_k = f(k)$ for all positive integers $k$, then

$$\sum_{k=1}^{\infty} a_k \text{ converges } \Longleftrightarrow \int_1^{\infty} f(x)\,dx \text{ converges.}$$

pf (see your textbook)

Ex. Use the integral test to determine whether the series: $\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{11}} + \cdots$ converges or diverges.

Solution

In "$\Sigma$" notation we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}}$

So $a_n = \frac{1}{\sqrt{2n+1}} \Rightarrow f(x) = \frac{1}{\sqrt{2x+1}}$ which is positive and nonincreasing on $[1, \infty)$ thus the integral test applies:

$$\int_1^{\infty} \frac{1}{\sqrt{2x+1}}\,dx = \lim_{b\to\infty} \int_1^b (2x+1)^{-1/2}\,dx = \lim_{b\to\infty} \sqrt{2x+1}\,\Big|_1^b$$

$$= \lim_{b\to\infty} \left(\sqrt{2b+1} - \sqrt{3}\right) = \infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}} \text{ diverges.}$$

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Def A series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$ where $p$ is a constant is called a p-series.

Thm p-series test (very important)

1) $p > 1 \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^p}$ converges.

2) $p \leq 1 \Rightarrow$ " diverges.

pf Use the integral test with $f(x) = \frac{1}{x^p}$. Note we solved this integral carefully in lecture notes for section 8.4, and the $p<0$ case diverges by the nth term test (section 9.2).

Ex

Does $\sum_{k=1}^{\infty} \frac{1}{k^{1.0001}}$ converge or diverge?

Solution: This is a p-series with $p = 1.0001 > 1$ thus it converges by the p-series test.

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Note: The question of convergence or divergence of a series is determined completely by the tail of the series. By this we mean that you may safely ignore any finite number of terms at the beginning of a series without affecting the convergence or divergence of the series as a whole.

Ignoring terms will change the sum if the series converges, but this is usually easy to compensate for.