§9.2 Infinite Series

Def An *infinite series* is a list of numbers separated by '+' symbols, with one number for each positive integer. (In other words, it's an infinite sum)

Note: From here on out I'll drop the "infinite" from infinite sequence and infinite series, and just refer to sequences and series, but it is still implied. (Finite series aren't interesting.)

Examples

1) $1+3+5+7+9+11+\cdots = \sum_{n=1}^{\infty} 2n-1$

2) $\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\frac{1}{5^4}+\cdots = \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$

3) $(-1)+1+(-1)+1+\cdots = \sum_{n=1}^{\infty} (-1)^n$

4) $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots = \sum_{n=1}^{\infty} \frac{1}{2n}$

5) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

There are two natural questions, that we care about:

1) Does the series converge or diverge?
(i.e. Does the series sum to a finite number or not?)

2) If a series converges, what number does it converge to?
(i.e. What number does the series sum to?)

To answer these questions, <u>we will associate a sequence with each series.</u> This sequence is called the <u>partial sum sequence</u>, and is denoted by $\{S_n\}$.

<u>Example</u>

$$1+3+5+7+9+11+\cdots = \sum_{n=1}^{\infty} 2n-1 = \sum_{n=1}^{\infty} a_n \quad \underline{\text{series}}$$

$$1, 4, 9, 16, 25, 36, \cdots = \{n^2\}_{n=1}^{\infty} = \{S_n\}_{n=1}^{\infty} \quad \underline{\text{sequence}}$$

$$S_1, S_2, S_3, S_4, S_5, S_6, \cdots$$

$$S_1 = 1$$
$$S_2 = 1+3 = 4$$
$$S_3 = 1+3+5 = 9$$
$$S_4 = 1+3+5+7 = 16$$
$$S_5 = 1+3+5+7+9 = 25$$
$$S_6 = 1+3+5+7+9+11 = 36$$
$$\vdots \qquad \ddots$$

This is how we generate the sequence of partial sums $\{S_n\}$.

$$S_n = \underbrace{1+3+5+7+\cdots+(2(n-1)-1)+(2n-1)}_{\text{open form}} = \underline{n^{th} \text{ partial sum}}$$

<u>open form</u> (a sum with "..." in it)

<u>Key Idea</u>

With the above series it is obvious to see that it doesn't sum to a finite number, but in general it won't be obvious! The key is to go from the <u>open form</u> of the <u>nth</u> partial sum to a <u>closed form</u> and then take the limit, as $n \to \infty$.

Def A <u>closed form</u> is a compact formula for finding the value of the n^{th} partial sum.

For the above example I wrote $\{S_n\} = \{n^2\}$, (closed form) but how do I know that for certain? The first six terms of $\{S_n\}$, that is S_1, S_2, S_3, S_4, S_5 & S_6 certainly seem to indicate that $S_n = n^2$, but how do we know for sure? How can we be certain that $S_n = n^2$ for all $n = 1, 2, 3, 4, 5, \ldots$?

We can't check them all!!

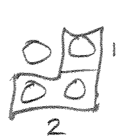
How do we know (for certain) that

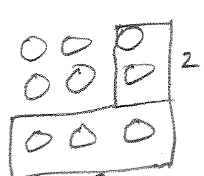
$$S_n = \underbrace{1+3+5+7+\cdots+(2(n-1)-1)+(2n-1)}_{\text{open form}} = \underset{\text{closed form}}{n^2} \;??$$

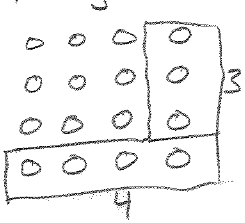
We must prove it!

Graphical proof

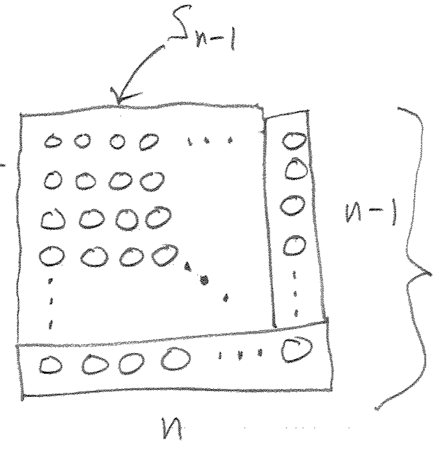
$n=1$ ○ $1 = 1^2 = S_1$

$n=2$  $1+3 = 2^2 = S_2$

$n=3$  $1+3+5 = 3^2 = S_3$

$n=4$  $1+3+5+7 = 4^2 = S_4$

<u>In general</u>



$\Rightarrow \boxed{S_n = n^2}$

$$\boxed{\begin{aligned} S_n &= S_{n-1} + (n-1) + n \\ &= S_{n-1} + (2n-1) \end{aligned}}$$

(cont.) →

Now that we have a closed form formula for S_n, i.e. a formula for the n^{th} partial sum, we can take the limit of S_n as $n \to \infty$ to see if the series, $\sum_{n=1}^{\infty} 2n-1$ converges or diverges.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n^2 = \infty \text{ or D.N.E.}$$

(does not exist)

Thus the series diverges.

Ex Does the series $\sum_{n=1}^{\infty} \frac{3}{10^n}$ converge or diverge? If it converges, what is its sum?

Solution This is a geometric series. A geometric series is a series where the ratio of any two consecutive terms in the series (sum) is constant.

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1{,}000} + \frac{3}{10{,}000} + \cdots = \sum_{n=1}^{\infty} 3 \cdot \left(\frac{1}{10}\right)^n$$

$$\frac{3/100}{3/10} = \frac{1}{10} \qquad \frac{3/10{,}000}{3/1{,}000} = \frac{1}{10}$$

Recall that our goal is to find a closed form formula for the n^{th} term of the partial sum sequence, and then take its limit.

$$S_n = \underbrace{\frac{3}{10} + \frac{3}{100} + \frac{3}{1{,}000} + \cdots \frac{3}{10^{n-1}} + \frac{3}{10^n}}_{\text{open form}}$$

(this is always finite)

(cont.) →

Here is the trick:

$$S_n = \frac{3}{10} + \cancel{\frac{3}{100}} + \cancel{\frac{3}{1{,}000}} + \cancel{\cdots} + \cancel{\frac{3}{10^{n-1}}} + \cancel{\frac{3}{10^n}}$$

$$-\ \frac{1}{10}S_n = \cancel{\frac{3}{100}} + \cancel{\frac{3}{1{,}000}} + \cancel{\frac{3}{10{,}000}} + \cancel{\cdots} + \cancel{\frac{3}{10^n}} + \frac{3}{10^{n+1}}$$

$$S_n - \frac{1}{10}S_n = \frac{3}{10} - \frac{3}{10^{n+1}}$$

$$\Rightarrow \left(1-\frac{1}{10}\right)S_n = \frac{3}{10}\left(1 - \frac{1}{10^n}\right)$$

$$\Rightarrow \quad \frac{9}{10}S_n = \frac{3}{10}\left(1 - \frac{1}{10^n}\right)$$

$$S_n = \frac{\cancel{10}}{9}\cdot\frac{3}{\cancel{10}}\left(1 - \frac{1}{10^n}\right)$$

$$S_n = \frac{1}{3}\left(1 - \frac{1}{10^n}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{3}\left(1 - \frac{1}{10^n}\right) = \boxed{\frac{1}{3}}$$

Thus $\sum_{n=1}^{\infty} \frac{3}{10^n} = \boxed{\frac{1}{3}}$ and thus converges.

Let's do the above procedure again for the general case of a geometric series: $\sum_{n=1}^{\infty} ar^{n-1} = a + ar^1 + ar^2 + ar^3 + \cdots$

$$S_N = a + \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots + \cancel{ar^{N-1}} = \sum_{n=1}^{N} ar^{n-1}$$

$$-\ rS_N = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots + \cancel{ar^{N-1}} + ar^N$$

$$(1-r)S_N = a - ar^N = a(1-r^N) \Rightarrow \boxed{S_N = \frac{a(1-r^N)}{(1-r)}} \rightarrow$$

Now that we have a closed form expression for the N^{th} partial sum in the sequence of partial sums, we can take its limit as $N \to \infty$ to determine whether the geometric series converges or diverges:

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a(1-r^N)}{(1-r)} = \lim_{N\to\infty} \left[\frac{a}{1-r} + \frac{-a}{1-r} r^N \right]$$

constants

Thus convergence or divergence is wholly determined by whether r^N converges or diverges as $N \to \infty$.

what a & r mean in the geometric series

$a =$ 1st term in series
$r =$ common ratio of consecutive terms

If $|r|<1$ then $\lim_{N\to\infty} \left[\frac{a}{1-r} + \frac{-a}{1-r} \cancel{r^N}^{0} \right] = \boxed{\frac{a}{1-r}}$ converges.

If $|r| \geq 1$ then the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

Example Find the sum of: $\frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \frac{4}{81} + \ldots$

$a = \frac{4}{3}$ $\quad r = \frac{1}{3}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{4}{3}\left(\frac{1}{3}\right)^{n-1} = \frac{4/3}{1-1/3} = \frac{4}{\cancel{3}} \cdot \frac{\cancel{3}}{2} = \boxed{2}$$

Q: what if $r = 1$? A: $\sum_{n=1}^{\infty} ar^{n-1} = a + a + a + \ldots = \begin{cases} \infty & a>0 \\ -\infty & a<0 \end{cases}$

The Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

The above series is called <u>the harmonic series</u>. Unlike arithmetic and geometric series, which are classes of infinite series, there is only one harmonic series.

<u>Q</u>: Does the harmonic series converge?

<u>A</u>: No. Despite the fact that the numbers in the series are getting closer and closer to zero, the sum is infinite!

<u>proof</u> There are many proofs of this fact. The proof we give now relies on making estimates. First we do some clever insertion of parentheses:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> 2 \cdot \frac{1}{4} = \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> 4 \cdot \frac{1}{8} = \frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right)}_{> 8 \cdot \frac{1}{16} = \frac{1}{2}} + \cdots$$

Thus by going out far enough in the series, we can add numbers greater than $\frac{1}{2}$ an infinite number of times and thus the series diverges. $\square$

<u>Thm</u> <u>n^{th} - Term Test for Divergence</u> (very important test)

<u>If</u> the series $\sum_{n=1}^{\infty} a_n$ converges,

<u>Then</u> $\lim_{n \to \infty} a_n = 0$.

<u>Remark 1</u>: The converse, i.e. (If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges) is <u>false</u> as the harmonic series illustrates.

$\rightarrow$

Remark 2: The contrapositive form of the statement in the theorem is more useful:

$$\boxed{\text{If } \lim_{n\to\infty} a_n \neq 0 \text{ Then } \sum_{n=1}^{\infty} a_n \text{ diverges.}}$$

statement	converse	Contrapositive
$p \Rightarrow q$	$q \Rightarrow p$	$\sim q \Rightarrow \sim p$

always equivalent

Ex Show that $\sum_{n=1}^{\infty} \frac{n^3}{3n^3+2n^2}$ diverges.

$$\lim_{n\to\infty} \frac{n^3}{3n^3+2n^2} \cdot \frac{1/n^3}{1/n^3} = \lim_{n\to\infty} \frac{1}{3+2/n} = \frac{1}{3} \neq 0.$$

Thus by the n^{th} term test, the series diverges.

When writing an answer on your exam, it is important to state which theorem you are using to reach your conclusion.

Collapsing Series (uses partial fraction decomposition)

Ex Show that the following series converges and find its sum.

$$\sum_{k=1}^{\infty} \frac{1}{(k+2)(k+3)}$$

First: $\frac{1}{(k+2)(k+3)} = \frac{A}{k+2} + \frac{B}{k+3} \Rightarrow 1 = A(k+3) + B(k+2)$

@ $k=-2$: $1 = A$

@ $k=-3$: $1 = -B \Rightarrow B = -1$

$$\boxed{\frac{1}{(k+2)(k+3)} = \frac{1}{k+2} + \frac{-1}{k+3}}$$

This will allow us to get a closed form for S_N →

$$\sum_{k=1}^{N} \frac{1}{k+2} + \frac{-1}{k+3} = \underbrace{\left(\frac{1}{3} + \cancel{\frac{-1}{4}}\right)}_{k=1} + \underbrace{\left(\cancel{\frac{1}{4}} + \cancel{\frac{-1}{5}}\right)}_{k=2} + \underbrace{\left(\cancel{\frac{1}{5}} + \frac{-1}{6}\right)}_{k=3} + \cdots$$

These terms will cancel too.

$$+ \underbrace{\left(\frac{1}{N} + \cancel{\frac{-1}{N+1}}\right)}_{k=N-2} + \underbrace{\left(\cancel{\frac{1}{N+1}} + \cancel{\frac{-1}{N+2}}\right)}_{k=N-1} + \underbrace{\left(\cancel{\frac{1}{N+2}} + \frac{-1}{N+3}\right)}_{k=N}$$

$$\Rightarrow S_N = \sum_{k=1}^{N} \frac{1}{(k+2)(k+3)} = \underbrace{\frac{1}{3} + \frac{-1}{N+3}}_{\text{closed form}}$$

$$\Rightarrow \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1}{3} + \frac{-1}{N+3} = \frac{1}{3} \Rightarrow \boxed{\sum_{k=1}^{\infty} \frac{1}{(k+2)(k+3)} = \frac{1}{3}}$$

Ex $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k(k+2)} = ?$ $\qquad \displaystyle\frac{2}{k(k+2)} = \frac{A}{k} + \frac{B}{k+2} \Rightarrow 2 = (k+2)A + kB$

@ $k=0$: $2 = 2A \Rightarrow \boxed{A=1}$

@ $k=-2$: $2 = -2B \Rightarrow \boxed{B=-1}$

$$\frac{2}{k(k+2)} = \frac{1}{k} + \frac{-1}{k+2}$$

$$S_N = \sum_{k=1}^{N} \frac{1}{k} + \frac{-1}{k+2} = \underbrace{\left(\frac{1}{1} + \cancel{\frac{-1}{3}}\right)}_{k=1} + \underbrace{\left(\frac{1}{2} + \cancel{\frac{-1}{4}}\right)}_{k=2} + \underbrace{\left(\cancel{\frac{1}{3}} + \cancel{\frac{-1}{5}}\right)}_{k=3} + \underbrace{\left(\cancel{\frac{1}{4}} + \boxed{\frac{-1}{6}}\right)}_{k=4} + \underbrace{\left(\cancel{\frac{1}{5}} + \boxed{\frac{-1}{7}}\right)}_{k=5} + \cdots$$

$$\underbrace{\left(\boxed{\frac{1}{N-4}} + \cancel{\frac{-1}{N-2}}\right)}_{k=N-4} + \underbrace{\left(\boxed{\frac{1}{N-3}} + \cancel{\frac{-1}{N-1}}\right)}_{k=N-3} + \underbrace{\left(\cancel{\frac{1}{N-2}} + \cancel{\frac{-1}{N}}\right)}_{k=N-2} + \underbrace{\left(\cancel{\frac{1}{N-1}} + \frac{-1}{N+1}\right)}_{k=N-1} + \underbrace{\left(\cancel{\frac{1}{N}} + \frac{-1}{N+2}\right)}_{k=N}$$

Note: the boxed terms cancel with terms in the "..."

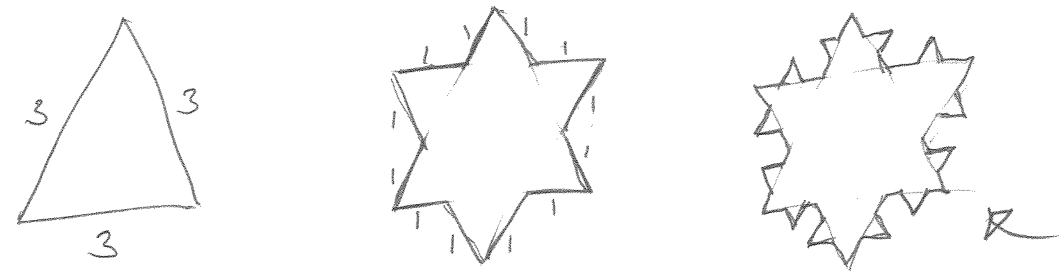
$$\Rightarrow S_N = 1 + \frac{1}{2} + \frac{-1}{N+1} + \frac{-1}{N+2} \quad \text{(closed form)}$$

$$\lim_{N\to\infty} S_N = \frac{3}{2} \Rightarrow \boxed{\sum_{k=1}^{\infty} \frac{2}{k(k+2)} = \frac{3}{2}}$$

The Koch Snowflake

- This is a fractal that is built from an equilateral triangle, by iterating a simple procedure.

$n=1$ $\qquad$ $n=2$ $\qquad$ $n=3$



each side has length $\frac{1}{3}$

$P=9$ $\qquad$ $P=12$ $\qquad$ $P=16$

The Koch snowflake is the result of repeating the above procedure infinitely many times.

- Q: What is the perimeter of the snowflake?

A: ∞.

The area is finite. (I'll let you figure out how to prove that).

We have a geometric figure with finite area, but infinite perimeter!?! WOW!

pf Perimeter = (# sides)(length of each side)

$$P_0 = (3)(3) \qquad P_n = (3\cdot 4^n)\cdot\left(\tfrac{1}{3}\right)^{n-1}\cdot\underbrace{\left(\tfrac{1}{3}\right)\cdot 3}_{=1}$$

$$P_1 = (3\cdot 4)\cdot(1) \qquad P_n = 3^2\cdot 4^n\cdot\left(\tfrac{1}{3}\right)^n$$

$$P_2 = (3\cdot 4^2)\cdot\left(\tfrac{1}{3}\right)$$

$$P_3 = (3\cdot 4^3)\cdot\left(\tfrac{1}{3^2}\right) \qquad \boxed{P_n = 9\cdot\left(\tfrac{4}{3}\right)^n} \qquad \boxed{\lim_{n\to\infty} P_n = \infty}$$

Area $= \frac{8}{5}$ (area of base triangle)