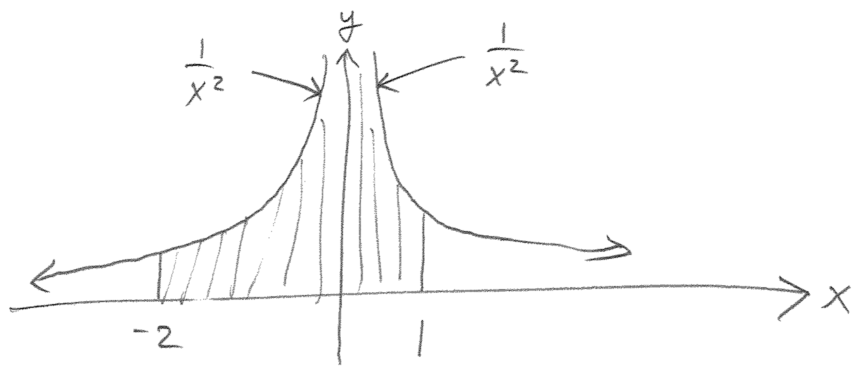
§ 8.4 Improper Integrals : Infinite Integrands

Consider the definite integral:

$$\int_{-2}^{1} \frac{1}{x^2}\,dx = \int_{-2}^{1} x^{-2}\,dx = -x^{-1}\Big|_{-2}^{1} = \frac{-1}{x}\Big|_{-2}^{1} = \frac{-1}{1} + \frac{1}{-2} = \boxed{\frac{-3}{2}} \times$$

This answer is actually wrong! Why?

The problem occurs as $x \to 0$. When $x \to 0$, the integrand "blows up", that is it becomes a huge positive number. Look at the graph of $\frac{1}{x^2}$ below and you will be able to see why our negative answer of $-\frac{3}{2}$ is absurd!



The area is all positive, but the integral is negative!

Whenever your integrand has a singularity (a point which causes the denominator to be zero) and you are computing a definite integral whose bounds of integration include the singularity then we say that you have an

"improper integral with an infinite integrand".

We solve the singularity problem, like we do every other problem by using our trusty tool the limit.

Def <u>If</u> f is continuous on $[a,b)$ (half-open interval)

and $\lim_{x \to b^-} f(x) = \pm\infty$

<u>Then</u> "is defined to be"

$$\int_a^b f(x)\,dx := \lim_{t \to b^-} \int_a^t f(x)\,dx$$

when the limit exists and is finite.

otherwise we say <u>the integral diverges.</u>

Here's how we use the definition above to resolve our problem:

$$\int_{-2}^{1} \frac{1}{x^2}\,dx = \int_{-2}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$

$$= \lim_{t \to 0^-} \int_{-2}^{t} \frac{1}{x^2}\,dx + \lim_{t \to 0^+} \int_{t}^{1} \frac{1}{x^2}\,dx$$

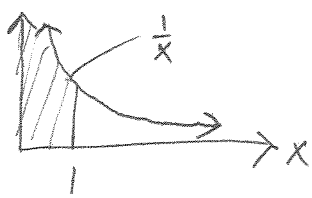
$$= \lim_{t \to 0^-} \left.\frac{-1}{x}\right|_{-2}^{t} + \lim_{t \to 0^+} \left.\frac{-1}{x}\right|_{t}^{1}$$

$$= \lim_{t \to 0^-} -\left(\frac{1}{t} + \frac{1}{2}\right) + \lim_{t \to 0^+} -\left(\frac{1}{1} - \frac{1}{t}\right)$$

$$= +\infty + \infty = \infty \implies \int_{-2}^{1} \frac{1}{x^2}\,dx \text{ diverges!}$$

Ex Evaluate, if possible

$$\int_0^1 \frac{1}{x}\,dx$$

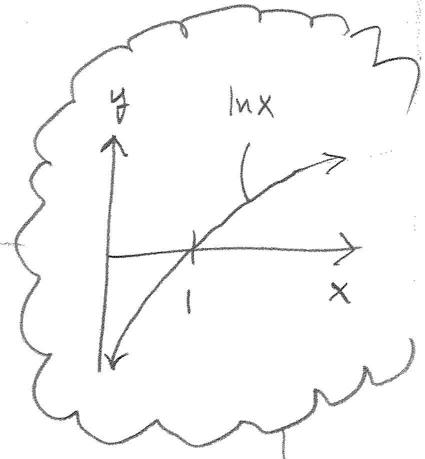


finite area?

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \ln x \Big|_t^1$$

$$= \lim_{t\to 0^+} \overset{0}{\cancel{\ln 1}} - \overset{-\infty}{\cancel{\ln t}} = +\infty$$

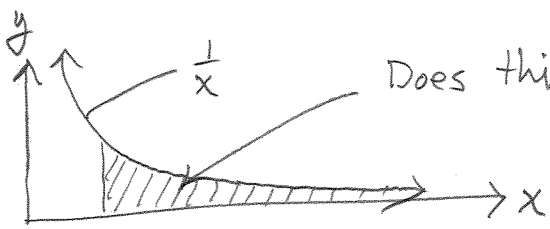
Not a number!



$$\Rightarrow \int_0^1 \frac{1}{x}\,dx \text{ diverges}$$ infinite area!

Ex. Let's go back to a problem from section 8.3 closely related to the previous example.

$$\int_1^\infty \frac{1}{x}\,dx$$



Does this area diverge too?

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x \Big|_1^b = \lim_{b\to\infty} \overset{\infty}{\cancel{\ln b}} - \overset{=0}{\ln 1}$$

Thus $\int_1^\infty \frac{1}{x}\,dx$ diverges infinite area!

Q: Is there a $p \in \mathbb{R}$ such that $\int_0^1 \frac{1}{x^p}\,dx$ or $\int_1^\infty \frac{1}{x^p}\,dx$ converge?

A: Yes, but not simultaneously!

Ex. $\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+} \int_t^1 x^{-p}\,dx \overset{p\neq 1}{=} \lim_{t\to 0^+} \left.\frac{x^{-p+1}}{-p+1}\right|_t^1$

$\boxed{p \neq 1}$

$\boxed{p>0}$ $= \lim_{t\to 0^+} \left[\frac{1}{1-p} - \frac{1}{1-p}\cdot\frac{1}{t^{p-1}}\right]$

2 cases to consider
$p>1$
$p<1$

$p>1$ ex $p=3$ $\lim_{t\to 0^+} \frac{1}{t^{3-1}} = \lim_{t\to 0^+} \frac{1}{t^2} = \boxed{\infty}$

$p<1$ ex $p=\frac{1}{2}$ $\lim_{t\to 0^+} \frac{1}{t^{\frac{1}{2}-1}} = \lim_{t\to 0^+} \frac{1}{t^{-1/2}}$

$= \lim_{t\to 0^+} \sqrt{t} = \boxed{0}$

Thus $\boxed{\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{1-p} & p<1 \\ \infty & p>1 \end{cases}}$

Ex. $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx \overset{\boxed{p\neq 1}}{=} \lim_{b\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^b$

$= \lim_{b\to\infty} \left(\frac{1}{1-p}\right)\left[\frac{1}{b^{p-1}} - 1\right]$

2 cases to consider
$p>1$
$p<1$

$p>1$: ex $p=3$ $\lim_{b\to\infty} \frac{1}{b^{3-1}} = \lim_{b\to\infty} \frac{1}{b^2} = \boxed{0}$

$p<1$: ex $p=\frac{1}{2}$ $\lim_{b\to\infty} \frac{1}{b^{\frac{1}{2}-1}} = \lim_{b\to\infty} b^{1/2} = \boxed{\infty}$

Thus $\boxed{\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \infty & p<1 \\ \frac{1}{p-1} & p>1 \end{cases}}$