§ 8.3 Improper Integrals : Infinite Limits of Integration

In many instances in Math, Science and Engineering (e.g. Fourier Transforms) one wishes to compute definite integrals where one or both of the bounds is infinite, e.g.

$$\int_0^\infty f(x)\,dx \quad \underline{\text{OR}} \quad \int_{-\infty}^1 f(x)\,dx \quad \text{OR} \quad \int_{-\infty}^\infty f(x)\,dx$$

Fourier Transform : $\hat{f}(s) = \int_{-\infty}^{\infty} f(x)\,e^{-2\pi i x}\,dx$

(You don't need to know the Fourier Transform, I just want you to be aware that it is extremely important and useful in all three disciplines mentioned above.)

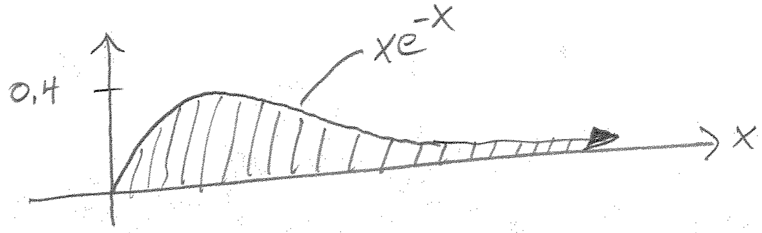
Definition (One limit infinite)

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx$$

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$$

We say the integral converges if the limit exists. Otherwise the integral diverges

Ex. $\int_0^\infty x e^{-x}\,dx = \lim_{b\to\infty} \int_0^b x e^{-x}\,dx$



As you can see by the graph, xe^{-x} decays to 0 rap ly. Thus it is likely that the limit exists.

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Integration by parts:

$$\lim_{b\to\infty}\int_0^b x e^{-x}\,dx$$

LIPET

$u = x \qquad \int dv = \int e^{-x}\,dx$

$du = dx \qquad v = -e^{-x}$

So, $$\lim_{b\to\infty}\int_0^b x e^{-x}\,dx = \lim_{b\to\infty} -xe^{-x}\Big|_0^b + \int_0^b e^{-x}\,dx$$

$$= \lim_{b\to\infty} -xe^{-x}\Big|_0^b - e^{-x}\Big|_0^b = \lim_{b\to\infty} -e^{-x}(1+x)\Big|_0^b$$

$$= \lim_{b\to\infty} \left[\cancel{-e^{-b}(1+b)}\right]^{0} + \underbrace{\left[e^{-0}(1+0)\right]}_{=1}$$

$$= \boxed{1}$$

Def (Both limits infinite)

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{b\to\infty}\int_0^b f(x)\,dx$$

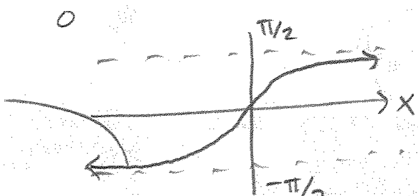
If both limits converge then $\int_{-\infty}^{\infty} f(x)\,dx$ converges, otherwise it diverges.

Ex. Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ or state that it diverges.

Solution $$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \tan^{-1}x\Big|_0^b$$

$$= \lim_{b\to\infty} \cancel{\tan^{-1}b}^{\,\pi/2} - \cancel{\tan^{-1}0} = \frac{\pi}{2}$$

$\tan^{-1}x$



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Notice: $\dfrac{1}{1+x^2}$

1 ← degree 0 poly. even

x^2 ← degree 2 poly even

1 ← degree 0 poly. even

$\Big\}$ sum of even functions is even

pf $f(x), g(x)$ even

$f(x)+g(x) = f(-x)+g(-x)$
$= f(x)+g(x)$

$\Rightarrow (f+g)(-x) = (f+g)(x)$ $\square$

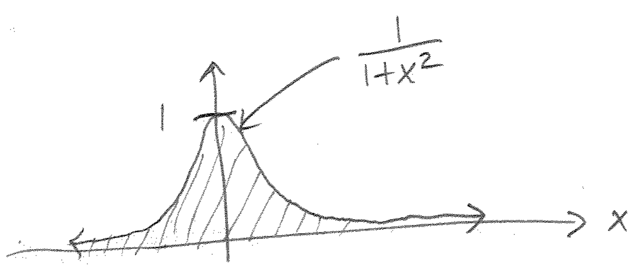
So we have

$$\frac{\text{even}}{\text{even}} \sim \frac{+1}{+1} = +1 \Rightarrow \text{even}$$

Thus $$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$$

" Exploit symmetry whenever you can! "

Thus $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}.$$



From the graph we see that $\frac{1}{1+x^2}$ is symmetric with respect to the y-axis and hence even.

Aside

Polynomials with only odd powers of x are odd,

e.g. $f(x) = x^5 - 3x^3 + x$

and polynomials with only even powers of x are even.

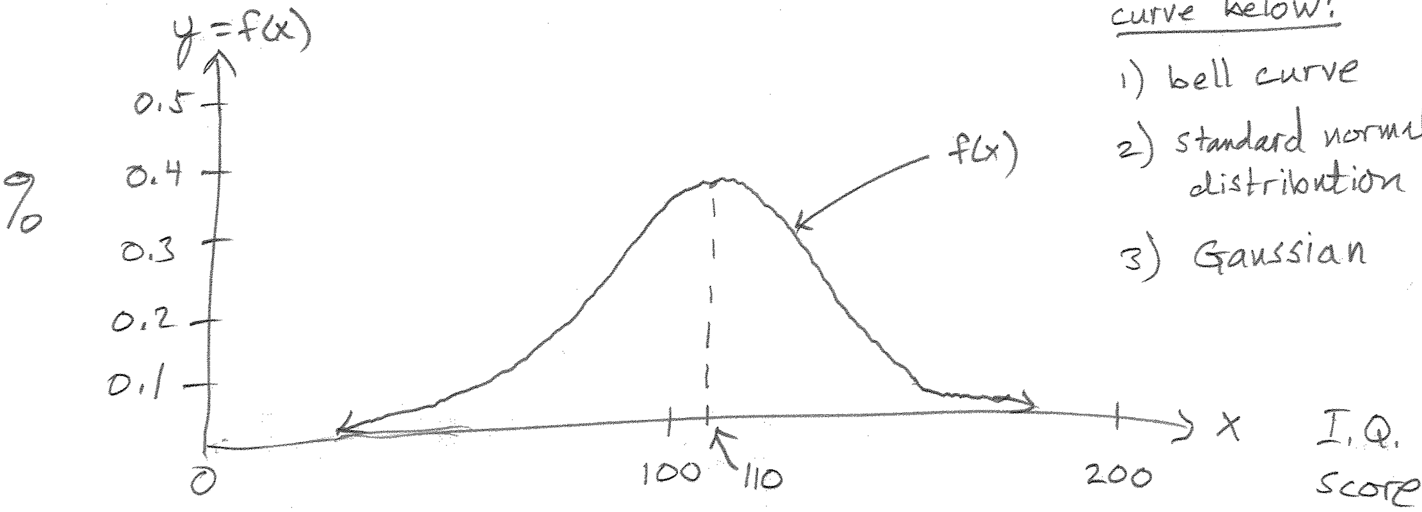
and polynomials with both even and odd powers of x are neither even nor odd.

Probability Density Functions

Def A random variable is simply a quantity that you don't know the value of until you measure it.

Def A probability density function or PDF comes from measuring a random variable many, many times and then plotting the measurements on the x-axis and the percentage of all measurements which correspond to that x value on the y-axis.

For example, suppose we administered an I.Q. test to 10,000 high school students in utah, then we could plot the pdf like so:



Names for the curve below:

1) bell curve
2) standard normal distribution
3) Gaussian

Two key properties of all PDF's:

① $f(x) \geq 0$

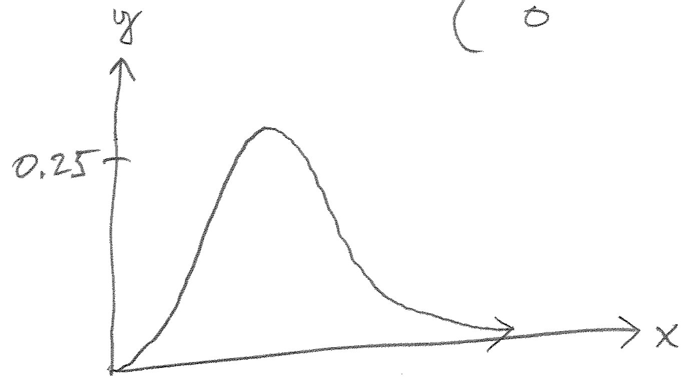
② $\int_{-\infty}^{\infty} f(x)\,dx = 1$ ⟹ Area under the curve is 1!

The material in these lecture notes is often skipped in Calculus I courses, so if it is new you might want to read section 5.7 in your text.

Ex The exponential distribution (a pdf often used to model the lifetime of mechanical or electrical components.)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & 0 \le x \\ 0 & x < 0 \end{cases}$$

λ = "lambda"



It's very similar in shape to the bell curve, but only has a long tail on one side.

a) Show that: $\int_{-\infty}^{\infty} f(x)\,dx = 1$,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= \lim_{b\to\infty} \int_{0}^{b} \lambda e^{-\lambda x}\,dx = \lim_{b\to\infty} -e^{-\lambda x}\Big|_{0}^{b} = \lim_{b\to\infty} -\left[\cancel{e^{-\lambda b}} - e^{0}\right]$$

$$= \boxed{1}$$

b) Find the mean: μ = "mu"

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{0} x \cdot 0\,dx + \int_{0}^{\infty} x \lambda e^{-\lambda x}\,dx$$

expectation

$$\mu = \lim_{b\to\infty} \int_0^b x\lambda e^{-\lambda x}\,dx \qquad \begin{array}{ll} u = x & dv = \lambda e^{-\lambda x}dx \\ du = dx & v = -e^{-\lambda x} \end{array}$$

$$= \lim_{b\to\infty} -xe^{-\lambda x}\Big|_0^b + \int_0^b e^{-\lambda x}\,dx$$

$$= \lim_{b\to\infty} \underbrace{-be^{-\lambda b}}_{\infty\cdot 0} + \cancel{0\cdot e^{-\lambda\cdot 0}} + \frac{-1}{\lambda}e^{-\lambda x}\Big|_0^b$$

$$= \lim_{b\to\infty} \underbrace{\frac{-b}{e^{\lambda b}}}_{\infty/\infty} - \frac{1}{\lambda}\left[\cancel{e^{-\lambda b}}^{\to 0} - \cancel{e^{0}}^{\to 1}\right]$$

$$\overset{H}{=} \lim_{b\to\infty} \cancel{\frac{-1}{\lambda e^{\lambda b}}}^{\to 0} + \frac{1}{\lambda} = \boxed{\frac{1}{\lambda}}$$

c) <u>Find the variance</u>: $\sigma^2 = E(x^2) - \mu^2$

Note standard deviation $= \sigma = \sqrt{\sigma^2} = \sqrt{\text{variance}}$

$$\sigma^2 = E(x^2) - \mu^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left(\frac{1}{\lambda}\right)^2$$

$$= \int_{-\infty}^{0} x^2\cdot 0\,dx + \int_0^\infty x^2\lambda e^{-\lambda x}\,dx - \left(\frac{1}{\lambda}\right)^2 \qquad \boxed{\begin{array}{ll} u = x^2 & dv = \lambda e^{-\lambda x}dx \\ du = 2x\,dx & v = -e^{-\lambda x} \end{array}}$$

Integration by parts $= \left[-x^2e^{-\lambda x}\right]_0^\infty - \int_0^\infty (-e^{-\lambda x})2x\,dx - \left(\frac{1}{\lambda}\right)^2$

$$= -0 + 0 + 2\int_0^\infty xe^{-\lambda x}\,dx - \left(\frac{1}{\lambda}\right)^2$$

almost μ

$$= \frac{2}{\lambda}\cdot\overset{\mu}{\frac{1}{\lambda}} - \left(\frac{1}{\lambda}\right)^2 = \boxed{\frac{1}{\lambda^2}}$$