§ 8.2 Other Indeterminate Forms

This section covers techniques for handling the following indeterminate forms in limits:

$$\boxed{\frac{\infty}{\infty}}, \quad \boxed{0\cdot\infty, \quad \infty-\infty}, \quad \boxed{0^0, \quad \infty^0, \quad 1^\infty}$$

It turns out that l'Hôpital's Rule applies to the first case above, but for all other forms above, we have to algebraically manipulate the expression until it is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and then we can apply l'Hôpital's Rule.

Ex. $\displaystyle\lim_{x\to\infty} \frac{x}{e^x}$ ← $\frac{\infty}{\infty}$ form

$$\overset{H}{=} \lim_{x\to\infty} \frac{1}{e^x} = \boxed{0}$$

Remark: If n is any positive integer then

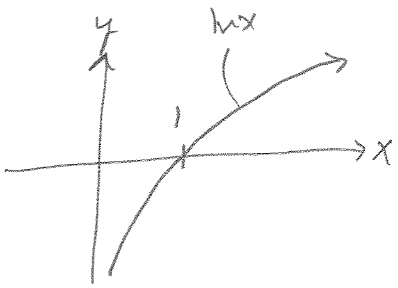
$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

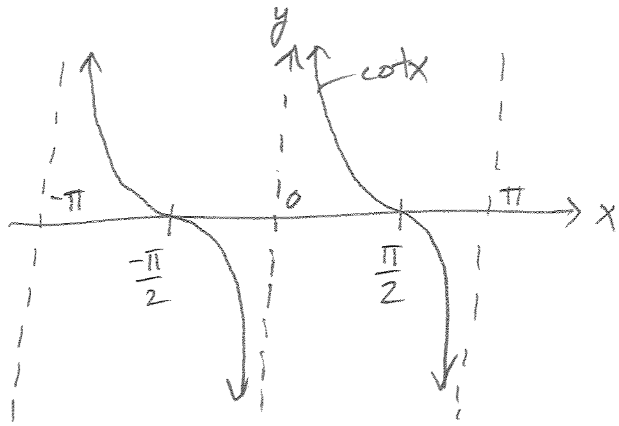
This is because after applying l'Hôpital's Rule n times, the numerator will be zero, but the denominator will still be e^x.

Ex $$\lim_{x\to\infty} \frac{x^3}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{3x^2}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{6x}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{6}{e^x} = 0$$

Ex

$$\lim_{x \to 0^+} \frac{\ln x}{\cot x} = \frac{-\infty}{\infty}$$





$$\lim_{x \to 0^+} \frac{\ln x}{\cot x} \overset{H}{=} \lim_{x \to 0^+} \frac{1/x}{-\csc^2 x}$$

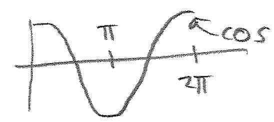
$$= -\lim_{x \to 0^+} \frac{\sin^2 x}{x} = -\lim_{x \to 0^+} \sin x \cdot \frac{\sin x}{x}$$

$$= -\left[\lim_{x \to 0^+} \sin x\right] \cdot \left[\lim_{x \to 0^+} \frac{\sin x}{x}\right]$$

$$= -0 \cdot 1 = \boxed{0}$$

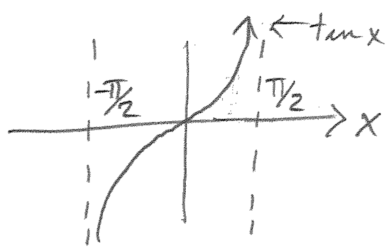
Ex

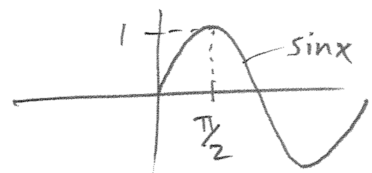
$$\boxed{0 \cdot \infty \text{ Form} \xRightarrow{\text{rewrite}} \frac{0}{0} \text{ Form}}$$



$$\lim_{x \to \frac{\pi}{2}} \left(\tan x \cdot \ln(\sin x)\right)$$

$$\infty \cdot 0$$





but $\ln(1) = 0$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\overset{1}{\sin x} \cdot \overset{0}{\ln(\sin x)}}{\underset{0}{\cos x}} \overset{H}{=} \lim_{x \to \frac{\pi}{2}} \frac{\overset{0}{\cos x} \cdot \overset{0}{\ln(\sin x)} + \cancel{\sin x} \cdot \frac{1}{\cancel{\sin x}} \cdot \cos x}{-\sin x} = \boxed{0}$$

Ex $\boxed{\infty - \infty \text{ Form} \xrightarrow{\text{rewrite}} \frac{0}{0} \text{ Form}}$

$$\lim_{x\to 1^+} \left(\underbrace{\frac{x}{x-1}}_{\to\infty} - \underbrace{\frac{1}{\ln x}}_{\to\infty} \right) = \lim_{x\to 1^+} \frac{x\ln x - x + 1}{(x-1)\ln x} = \frac{0}{0} \text{ form}$$

$$\overset{H}{=} \lim_{x\to 1^+} \frac{\ln x + x\cdot\frac{1}{x} - 1}{\ln x + (x-1)\cdot\frac{1}{x}}$$

$$= \lim_{x\to 1^+} \frac{\ln x}{\ln x + \frac{x-1}{x}} = \frac{0}{0} \text{ form}$$

$$\overset{H}{=} \lim_{x\to 1^+} \frac{1/x}{1/x + \frac{x-(x-1)}{x^2}}$$

$$= \lim_{x\to 1^+} \frac{1/x}{1/x + (1/x)^2} = \boxed{\frac{1}{2}}$$

When we have $\boxed{0^0,\ \infty^0,\ 1^\infty}$ the trick is to use logarithms:

Ex. $\lim_{x\to 0^+} (x+1)^{\cot x} = 1^\infty$ form

Set $y = (x+1)^{\cot x} \Rightarrow \ln y = \cot x \ln(x+1) = \frac{\ln(x+1)}{\tan x} = \frac{0}{0}$

$$\Rightarrow \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(x+1)}{\tan x} \overset{H}{=} \lim_{x\to 0^+} \frac{\overbrace{1/(x+1)}^{\to 1}}{\underbrace{\sec^2 x}_{\to 1}} = 1 = \lim_{x\to 0^+} \ln y$$

$$\Rightarrow \lim_{x\to 0^+} y = e^1 = \boxed{e}.$$

Ex $\displaystyle\lim_{x\to\pi/2^-} (\tan x)^{\cos x}$ $\boxed{\infty^0 \text{ form}}$

Let $y = \tan x^{\cos x} \implies \ln y = (\cos x)\ln(\tan x)$

$$= \frac{\ln(\tan x)}{\sec x} \quad \boxed{\frac{\infty}{\infty} \text{ form}}$$

$$\lim_{x\to\pi/2^-} \ln y = \lim_{x\to\pi/2^-} \frac{\ln(\tan x)}{\sec x} \overset{H}{=} \lim_{x\to\pi/2^-} \frac{\frac{\sec^2 x}{\tan x}}{\sec x \tan x}$$

$$= \lim_{x\to\pi/2^-} \frac{\sec x}{\tan^2 x}$$

$$= \lim_{x\to\pi/2^-} \frac{\cos x}{\sin^2 x} = \boxed{0}$$