§ 8.1 Indeterminate Forms of Type $\frac{0}{0}$

We now want to expand our repetoire of limits that we can evaluate. This will be helpful when we study the convergence of series in chapter nine.

l'Hôpital's Theorem

If $\lim_{x\to u} f(x) = 0$ and $\lim_{x\to u} g(x) = 0$ and

if $\lim_{x\to u} \frac{f'(x)}{g'(x)}$ exists, i.e. is either finite or infinite,

then $$\lim_{x\to u} \frac{f(x)}{g(x)} = \lim_{x\to u} \frac{f'(x)}{g'(x)}.$$

Ex. $$\lim_{x\to 0} \frac{\sin x}{x} \overset{H}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1$$

Ex. $$\lim_{x\to 3} \frac{x^2-9}{x^2-x-6} \overset{H}{=} \lim_{x\to 3} \frac{2x}{2x-1} = \frac{6}{5}$$

Ex. $$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \overset{H}{=} \lim_{x\to a} \frac{f'(x)}{1} = f'(a)$$

Remark We do not use the quotient rule to take the derivative of $\frac{f(x)}{g(x)}$ when applying l'Hôpital's theorem.

Instead we take two separate derivatives and look at their ratio.

Ex.

$$\lim_{x\to 0} \frac{1-\cos x}{x} \overset{H}{=} \lim_{x\to 0} \frac{\sin x}{1} = \boxed{0}$$

Ex. $$\lim_{x\to 0} \frac{\tan(2x)}{\ln(1+x)} \overset{H}{=} \lim_{x\to 0} \frac{2\sec^2(2x)}{1/(1+x)} = \frac{2}{1} = \boxed{2}$$

Note You can use l'Hôpital's Theorem repeatedly, as long as it applies, i.e. as long as you have the indeterminate form $\frac{0}{0}$.

Ex. $$\lim_{x\to 0} \frac{\sin x - x}{x^3} \overset{H}{=} \lim_{x\to 0} \frac{\cos x - 1}{3x^2}$$

$$\overset{H}{=} \lim_{x\to 0} \frac{-\sin x}{6x}$$

$$\overset{H}{=} \lim_{x\to 0} \frac{-\cos x}{6} = \boxed{\frac{-1}{6}}$$

Be Careful

Ex. $$\lim_{x\to 0} \frac{1-\cos x}{x^2+3x} \overset{H}{=} \lim_{x\to 0} \frac{\sin x}{2x+3} \overset{H}{=} \lim_{x\to 0} \frac{\cos x}{2} = \boxed{\frac{1}{2}}$$ WRONG

$\frac{0}{3}$

It is a mistake to apply l'Hôpital's thm. here.

So $$\lim_{x\to 0} \frac{1-\cos x}{x^2+3x} \overset{H}{=} \lim_{x\to 0} \frac{\sin x}{2x+3} = \frac{0}{3} = \boxed{0}$$ RIGHT

18.) $\displaystyle \lim_{x\to 0} \frac{e^x - \ln(1+x) - 1}{x^2} \overset{H}{=} \lim_{x\to 0} \frac{e^x + \frac{-1}{1+x}}{2x}$

$\frac{0}{0}$ $\qquad$ $\frac{0}{0}$

$\displaystyle \overset{H}{=} \lim_{x\to 0} \frac{e^x + \frac{1}{(1+x)^2}}{2}$

$\displaystyle = \frac{1+1}{2} = \boxed{1}$

24.) $\displaystyle \lim_{x\to 0^+} \frac{\int_0^x \sqrt{t}\cos t\,dt}{x^2} \overset{H}{=} \lim_{x\to 0^+} \frac{\sqrt{x}\cos x}{2x}$

$\displaystyle \overset{H}{=} \lim_{x\to 0^+} \left(\frac{\overset{\nearrow\infty}{\frac{\cos x}{2\sqrt{x}}}}{2} - \frac{\sqrt{x}\sin x}{2}\overset{\nearrow 0}{} \right)$

$= \boxed{\infty}$

$D_x\left[\sqrt{x}\cdot\cos x\right] = \frac{1}{2}x^{-1/2}\cdot\cos x + \sqrt{x}(-\sin x)$

$\displaystyle = \frac{\cos x}{2\sqrt{x}} - \sqrt{x}\sin x$

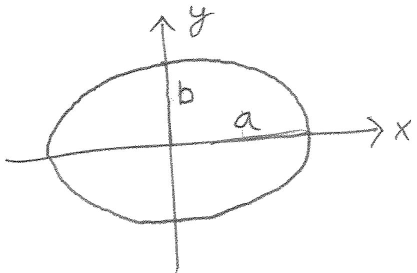
$D_x[2x] = 2$

Problem 31 from your text

Ex

$$\boxed{A = 2\pi b^2 + 2\pi a b\left[\frac{a}{\sqrt{a^2-b^2}}\sin^{-1}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\right]}$$

The above formula is for the surface area of a prolate spheroid, which is the surface of revolution created by rotating the ellipse given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ around the x-axis.



If $a>b$ and if we shrink "a" down until it is the same length as "b" then we get a circle of radius "b". Thus $\boxed{\lim_{a\to b^+} A = 4\pi b^2}$ or the area of a sphere.

$$\lim_{a\to b^+} A = 2\pi b^2 + 2\pi b \lim_{a\to b^+}\left(\frac{a^2\sin^{-1}\left(\frac{\sqrt{a^2-b^2}}{a}\right)}{\sqrt{a^2-b^2}}\right) \qquad \boxed{\frac{0}{0}\text{ form}}$$

$$\overset{H}{=} 2\pi b^2 + 2\pi b \lim_{a\to b^+}\frac{2a\sin^{-1}\left(\frac{\sqrt{a^2-b^2}}{a}\right)+a^2\frac{b}{a\sqrt{a^2-b^2}}}{\frac{a}{\sqrt{a^2-b^2}}}$$

Extra Credit

2 pts towards any exam (including the final)!

Show that the above circled limit is "b".

You must start with the circled part and apply l'Hôpital's Rule!