§7.5 Partial Fraction Decomposition

Recall that an improper fraction is a fraction of the form $\frac{a}{b}$ where $a > b$, e.g. $\frac{10}{3} = 3 + \frac{1}{3}$.

As shown above, we can always rewrite an improper fraction as a whole number plus a proper fraction, That is a fraction of the form $\frac{a}{b}$ where $a < b$, e.g. $\frac{2}{3}$.

Since we have a division algorithm for polynomials, we can do the same thing for rational functions.

A rational function is a ratio of two polynomials, e.g.

$$\frac{f(x)}{g(x)} = \frac{4x^3 + 3x + 1}{2x + 1}$$

when $\deg f > \deg g$ then we have an improper rational function. So the above example is an improper rational function. We use polynomial long division to change improper rat'l fcn's into a polynomial + a proper rat'l fcn.

Ex.

$$\begin{array}{r} 2x^2 - x + 2 + \frac{-1}{2x+1} \\ 2x+1 \,\big)\, 4x^3 + 0x^2 + 3x + 1 \\ -(4x^3 + 2x^2) \\ \hline -2x^2 + 3x \\ -(-2x^2 - x) \\ \hline 4x + 1 \\ -(4x + 2) \\ \hline -1 \end{array}$$

So:

$$\boxed{4x^3 + 3x + 1 = 2x^2 - x + 2 + \frac{-1}{2x+1}}$$

Q: why do this?

A: So we can integrate rational functions.

Ex.

$$\int \frac{4x^3+3x+1}{2x+1}\,dx = \int\left(2x^2-x+2+\frac{-1}{2x+1}\right)dx$$

$$= \frac{2}{3}x^3-\frac{1}{2}x^2+2x-\int\frac{1}{2x+1}\,dx$$

let $u = 2x+1$ $\quad du = 2dx \Rightarrow \frac{1}{2}du = dx$

$$= \frac{2}{3}x^3-\frac{1}{2}x^2+2x-\frac{1}{2}\int\frac{1}{u}\,du$$

$$= \boxed{\frac{2}{3}x^3-\frac{1}{2}x^2+2x-\frac{1}{2}\ln|2x+1|+C}$$

You already know how to integrate polynomials, so we will focus on 4 different types of remainder terms. The remainder in the above example is the simplest case, but the devil is in the details.

Case 1: Distinct Linear Factors (in denominator)

Ex $\quad \dfrac{2x+5}{x^2-6x+5} = \dfrac{2x+5}{(x-5)(x-1)} = \dfrac{A}{x-5}+\dfrac{B}{x-1}$

$\Rightarrow 2x+5 = A(x-1)+B(x-5) = Ax-A+Bx-5B = (A+B)x+(-A-5B)$

$\longrightarrow$

We have:

$$\underbrace{2x}+\underbrace{5} = \underbrace{(A+B)x} + \underbrace{(-A-5B)}$$

Equate the coefficients:

$$\left.\begin{aligned} A+B &= 2 \\ -A-5B &= 5 \end{aligned}\right\}$$ Linear system of two equations in two unknowns

Solve by elimination (i.e. adding a multiple of one equation to the other.) In this case, no multiple is necessary.

$$\begin{aligned} A+B &= 2 \\ -A-5B &= 5 \\ \hline 0 - 4B &= 7 \end{aligned} \implies \boxed{B = -\frac{7}{4}}$$

Thus $A + -\frac{7}{4} = 2 \implies A = 2 + \frac{7}{4} = \frac{8}{4} + \frac{7}{4} = \boxed{\frac{15}{4}}$ A

$$\implies \underbrace{\frac{2x+5}{(x-5)(x-1)}}_{\text{can't integrate directly}} = \underbrace{\frac{15/4}{x-5}} + \underbrace{\frac{-7/4}{x-1}}$$

can integrate directly

Shortcut for finding A & B: Evaluate at roots!

$$2x+5 = A(x-1) + B(x-5)$$

@ $x=1 \Rightarrow 2+5 = B(1-5)$

$$7 = -4B$$

$$\boxed{B = -7/4}$$

@ $x=5 \Rightarrow 10+5 = A(5-1)$

$$15 = 4A$$

$$\boxed{A = 15/4}$$

Case 2: Repeated Linear Factors

$$\frac{p(x)}{(x-r_1)^n (x-r_2)^m} = \frac{A_1}{x-r_1} + \frac{A_2}{(x-r_1)^2} + \cdots + \frac{A_n}{(x-r_1)^n} + \frac{B_1}{x-r_2} + \frac{B_2}{(x-r_2)^2} + \cdots + \frac{B_m}{(x-r_2)^m}$$

Ex. $\frac{1}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$

$\Rightarrow 1 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$

Use the evaluate at roots shortcut!

@ $x=1$: $1 = 2B \Rightarrow \boxed{B = \frac{1}{2}}$

@ $x=-1$: $1 = 4C \Rightarrow \boxed{C = \frac{1}{4}}$

@ $x=0$: $1 = -A + B + C \Rightarrow 1 = -A + \frac{1}{2} + \frac{1}{4}$

$\Rightarrow \boxed{A = -\frac{1}{4}}$

0 is not a root. Just a convenient value to allow us to solve for A!

→

So $\displaystyle\int \frac{1}{(x-1)^2(x+1)}\,dx = \int \left(\frac{-1/4}{x-1} + \frac{1/2}{(x-1)^2} + \frac{1/4}{x+1} \right) dx$

Let $u = x-1$; $du = dx$

$$= -\frac{1}{4}\ln|x-1| + \frac{1}{2}\int u^{-2}\,du + \frac{1}{4}\ln|x+1|$$

$$= \boxed{-\frac{1}{4}\ln|x-1| + \frac{1}{2}\left(\frac{-1}{x-1}\right) + \frac{1}{4}\ln|x+1| + C}$$

Case 3: Quadratic Factors

Write $\displaystyle\frac{p(x)}{(x^2+bx+c)\cdots} = \frac{Ax+B}{x^2+bx+c} + \cdots$ & cross-multiply.

Ex.

$$\frac{x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow x = A(x^2+1) + (Bx+C)(x-1)$$

$$= Ax^2 + A + Bx^2 - Bx + Cx - C$$

$$= (A+B)x^2 + (C-B)x + (A-C)$$

$$\Rightarrow 0 \cdot x^2 + 1 \cdot x + 0 = (A+B)x^2 + (C-B)x + (A-C)$$

→

So $A+B=0$ (1) ←
$-B+C=1$ (2) ←
$A-C=0$ (3) ←

equation #s

Eliminate B by adding equations (1) + (2) = (4):

$A+C=1$ (4)

Now add (4) & (3):

$A-C=0$ (3)

$2A+0C=1 \implies \boxed{A=\frac{1}{2}}$

$(1) \implies \frac{1}{2}+B=0 \implies \boxed{B=\frac{-1}{2}}$

$(3) \implies \frac{1}{2}-C=0 \implies \boxed{C=\frac{1}{2}}$

Thus,

$$\int \frac{x}{(x-1)(x^2+1)}\,dx = \int \left(\frac{1/2}{x-1} + \frac{-1/2\,x + 1/2}{x^2+1} \right) dx$$

$$= \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{x-1}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\left[\int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx\right]$$

$u = x^2+1$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

to
$x = 1\cdot\tan t$
$dx = \sec^2 t \cdot dt$
$t = \tan^{-1}(x)$

$$= \boxed{\frac{1}{2}\ln|x-1| - \frac{1}{2}\cdot\frac{1}{2}\ln(x^2+1) + \frac{1}{2}\tan^{-1}(x) + C}$$

Case 4: Repeated Quadratic Factors

If the rational function has the form:

$$\frac{p(x)}{(x^2+bx+c)^n} = \frac{A_1x+b_1}{x^2+bx+c} + \frac{A_2x+b_2}{(x^2+bx+c)^2} + \frac{A_3x+b_3}{(x^2+bx+c)^3} + \dots + \frac{A_nx+b_n}{(x^2+bx+c)^n}$$

Ex. $\dfrac{x^2+5x-1}{(x^2+1)^3(x^2+2x+4)^2}$

$$= \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+1)^2} + \frac{Ex+F}{(x^2+1)^3} + \frac{Gx+H}{(x^2+2x+4)} + \frac{Ix+J}{(x^2+2x+4)^2}$$

It would be heinously difficult to solve for $A, \dots, J$ above, but computers can do it pretty quickly.

Case 5: Repeated Linear Factors and Repeated Quadratic Factors

Ex.

$$\frac{x^2+3x}{(x^2+x+1)^2(x+1)^2} = \frac{Ax+B}{(x^2+x+1)} + \frac{Cx+D}{(x^2+x+1)^2} + \frac{E}{(x+1)} + \frac{F}{(x+1)^2}$$

$$= \frac{3x+2}{(x^2+x+1)} + \frac{x+3}{(x^2+x+1)^2} + \frac{-3}{(x+1)} + \frac{-2}{(x+1)^2}$$