# §7.4 Rationalizing Substitutions

This lesson will cover a new way of solving integrals of the type we saw in section 6.8:

1) $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

2) $\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

3) $\int \frac{1}{x\sqrt{x^2 - a^2}}\, dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$

But the method actually allows us to solve <u>many</u> more integrals than just the three listed above!

If your integral has any of the following expressions in the integrand then use the corresponding substitution:

| Radical | Substitution | Restriction on t |
|---|---|---|
| $\sqrt{a^2 - x^2}$ | $x = a\sin t$ | $-\pi/2 \le t \le \pi/2$ |
| $\sqrt{a^2 + x^2}$ | $x = a\tan t$ | $-\pi/2 < t < \pi/2$ |
| $\sqrt{x^2 - a^2}$ | $x = a\sec t$ | $0 \le t \le \pi$ $\left(t \ne \frac{\pi}{2}\right)$ |

Ex. Let's demonstrate this new technique on a problem we've previously solved. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$.

$\int \frac{1}{\sqrt{1-x^2}}\,dx$ let $x = \sin t$ $dx = \cos t\,dt$

$= \int \frac{1}{\sqrt{1-\sin^2 t}} \cdot \cos t\,dt$

Although at first glance this appears more complicated, it's actually a very simple expression in disguise!

Recall:

$\sin^2 t + \cos^2 t = 1$

$\Rightarrow \cos^2 t = 1 - \sin^2 t$

$= \int \frac{1}{\sqrt{\cos^2 t}} \cos t\,dt = \int \frac{\cancel{\cos t}}{\cancel{\cos t}}\,dt = \int dt = t + C$

Now, we want to go back to our original variable:

$x = \sin t \Longleftrightarrow \sin^{-1}x = t$ thus

$\boxed{\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C,}$ as before.

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This method is very versatile. It even works in situations without square roots.

$\boxed{\tan^2 t + 1 = \sec^2 t}$

Ex. $\int \frac{dx}{x^2+1}$ let $x = \tan t$ $dx = \sec^2 t\,dt$

$= \int \frac{1}{\tan^2 t + 1} \cdot \sec^2 t\,dt$ Recall $\frac{\sin^2 t + \cos^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$

$= \int \frac{\sec^2 t}{\sec^2 t}\,dt = \int dt = t + C = \boxed{\tan^{-1}x + C}$

Ex. $\int \sqrt{a^2 - x^2}\, dx$

$x = a\sin t \qquad dx = a\cos t\, dt$

$$= \int \sqrt{a^2 - a^2\sin^2 t}\; a\cos t\, dt$$

$$= \int a\sqrt{1 - \sin^2 t}\; a\cos t\, dt$$

$$= a^2 \int \sqrt{\cos^2 t}\; \cos t\, dt$$

Recall: Half-Angle Formula

$\cos^2 t = \frac{1}{2}(1 + \cos 2t)$

$$= a^2 \int \cos^2 t\, dt$$

$$= a^2 \int \frac{1}{2}(1 + \cos 2t)\, dt$$

$$= \frac{a^2}{2}\left[\int dt + \int \cos 2t\, dt\right]$$

$u = 2t$

$\frac{1}{2} du = dt$

$$= \frac{a^2}{2}\left[t + \frac{1}{2}\sin 2t\right] + C$$

Recall: Addition Formula

$\sin(s+t) = \sin s \cos t + \sin t \cos s$

So $\sin(t+t) = 2\sin t\cos t$

$$= \frac{a^2}{2}\left[t + \sin t\cos t\right] + C$$

We are not done! We must translate back from $t$ to $x$.

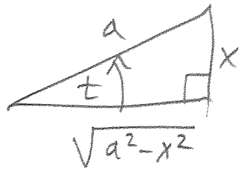
$$\boxed{\begin{array}{l} x = a\sin t \iff \sin^{-1}\left(\frac{x}{a}\right) = t \\ \Updownarrow \\ \sin t = \frac{x}{a} \end{array}}$$

$$= \frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\cos t\right] + C$$

still not done.

$$\cos t = \cos\Big(\underbrace{\sin^{-1}\left(\tfrac{x}{a}\right)}_{t}\Big)$$

$$\sin t = \frac{x}{a}$$




$$\Longrightarrow \cos t = \frac{\sqrt{a^2-x^2}}{a}$$

$$\therefore \int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{a^2}\right] + C$$

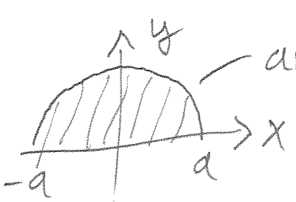
$$= \boxed{\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + C}$$

Recall the equation of a circle of radius a:

$$x^2 + y^2 = a^2 \Longrightarrow y^2 = a^2 - x^2$$

$$y = \pm\sqrt{a^2-x^2}$$

Top half of the circle is given by $y = +\sqrt{a^2-x^2}$




$$\text{area} = \int_{-a}^{a}\sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\Big|_{-a}^{a} + \cancel{\frac{x}{2}\sqrt{a^2-x^2}\Big|_{-a}^{a}} \rightarrow 0$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2} - \left(\frac{-\pi}{2}\right)\right)$$

$$= \boxed{\frac{a^2\pi}{2}} = \frac{1}{2}\text{ area of circle of radius } a!$$

Ex. what about more complicated things?

$$\int \frac{dx}{(x^2+4)^{3/2}}$$

Try $\boxed{x = \tan t}$ then we get $\tan^2 + 4$ in the denominator. We need a 1 here, so we can use a variation of the Pythagorean Identity $\sin^2 t + \cos^2 t = 1$.

Try $\boxed{x = 2\tan t}$ (why because $x^2 + 4 = x^2 + 2^2 = x^2 + a^2$.)

then $dx = 2\sec^2 t\, dt$

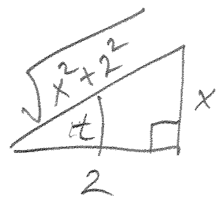
$$\int \frac{dx}{(x^2+4)^{3/2}} = \int \frac{1}{(2^2 \tan^2 + 2^2)^{3/2}} \cdot 2\sec^2 t\, dt$$

$$= \int \frac{1}{2^3(\tan^2 + 1)^{3/2}} \cdot 2\sec^2 t\, dt$$

$$= \frac{1}{2^2} \int \frac{\sec^2 t}{(\sec^2 t)^{3/2}}\, dt = \frac{1}{4} \int \frac{1}{\sec t}\, dt = \frac{1}{4} \int \cos t\, dt$$

$$= \frac{1}{4} \sin t + C$$

$\tan t = \frac{x}{2} \Rightarrow$




$\Rightarrow \sin t = \frac{x}{\sqrt{x^2+4}}$

$$= \boxed{\frac{1}{4} \frac{x}{\sqrt{x^2+4}} + C}$$