§7.3 Trigonometric Integrals

There are five common cases:

1. $\int \sin^n x\,dx$, $\int \cos^n x\,dx$
2. $\int \sin^m x\,dx \cos^n x\,dx$
3. $\int \sin mx \cos nx\,dx$, $\int \sin mx \sin nx\,dx$, $\int \cos mx \cos nx\,dx$
4. $\int \tan^n x\,dx$, $\int \cot^n x\,dx$
5. $\int \tan^m x \sec^n x\,dx$, $\int \cot^m x\,dx \csc^n x\,dx$

Type 1 a) n odd

$$\int \sin^5 x\,dx = \int \sin^4 x \sin x\,dx$$

Recall the Pythagorean Identity: $\cos^2 x + \sin^2 x = 1$

$\Rightarrow \sin^2 x = 1 - \cos^2 x$

$$\Rightarrow = \int (1-\cos^2 x)^2 \sin x\,dx$$

$$= \int (1 - 2\cos^2 x + \cos^4 x)\sin x\,dx$$

$$= \int \sin x\,dx - 2\int \cos^2 x \sin x\,dx + \int \cos^4 x \sin x\,dx$$

let $u = \cos x$ $-du = \sin x\,dx$

$$= -\cos x + 2\int u^2\,du - \int u^4\,du$$

$$= -\cos x + 2\frac{u^3}{3} - \frac{u^5}{5} + C$$

$$\boxed{= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C}$$

Type 1 b) <u>n even</u>

Recall the half-angle identities: $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$$\int \cos^4 x\, dx = \int \left[\tfrac{1}{2}(1+\cos 2x)\right]^2 dx$$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{4}\int 1\cdot dx + \frac{1}{4}\int 2\cos 2x\, dx + \frac{1}{4}\int \cos^2 2x\, dx$$

$$= \frac{1}{4}\int dx + \frac{1}{4}\int 2\cos 2x\, dx + \frac{1}{4}\int \frac{1}{2}(1+\cos 4x)\, dx$$

$$= \frac{x}{4} + \frac{1}{4}\sin 2x + \frac{1}{8}\int 1\cdot dx + \frac{1}{8}\int \cos 4x\, dx$$

$$= \text{''} + \text{''} + \frac{x}{8} + \frac{1}{8}\cdot\frac{1}{4}\int 4\cos 4x\, dx$$

$$= \text{''} + \text{''} + \text{''} + \frac{1}{32}\sin 4x + C$$

$$= \frac{x}{4} + \frac{x}{8} + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

$$= \boxed{\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C}$$

Type 2 a) m or n odd

$$\int \sin^3 x \cos^{-4} x \, dx = \int \sin^2 x \cos^{-4} x \sin x \, dx$$

$$= \int (1-\cos^2 x) \cos^{-4} x \sin x \, dx$$

$$= \int \cos^{-4} x \sin x \, dx - \int \cos^{-2} x \sin x \, dx$$

$u = \cos x$ $\quad du = -\sin x \, dx$

$$= -\int u^{-4} \, du + \int u^{-2} \, du$$

$$= -\left(\frac{-1}{3}\right) u^{-3} + (-1) u^{-1} + C$$

$$= \frac{1}{3} \cos^{-3} x - (\cos x)^{-1} + C$$

Note: $\cos^{-1} x \neq (\cos x)^{-1}$

$\cos^{-1} x = \arccos x$

$$= \boxed{\frac{1}{3} \sec^3 x - \sec x + C}$$

Type 2 b) both m and n even

$$\int \sin^2 x \cos^4 x \, dx = \int \tfrac{1}{2}(1-\cos 2x)\left[\tfrac{1}{2}(1+\cos 2x)\right]^2 dx$$

$$= \frac{1}{8}\int (1-\cos 2x)(1+2\cos 2x+\cos^2 2x)\,dx$$

$$= \frac{1}{8}\int\Big(1+2\cos 2x+\cos^2 2x + {}$$
$$- \cos 2x - 2\cos^2 2x - \cos^3 2x\Big)\,dx$$

$$= \frac{1}{8}\int\left(1+\cos 2x-\cos^2 2x-\cos^3 2x\right)dx$$

$$= \frac{1}{8}\left[\int 1\cdot dx + \int \cos 2x\,dx - \int \cos^2 2x\,dx - \int \cos^3 2x\,dx\right]$$

$$= \frac{1}{8}\left[\int dx + \int \cos 2x\,dx - \int \tfrac{1}{2}(1+\cos 4x)\,dx - \int (1-\sin^2 2x)\cos 2x\,dx\right]$$

$$= \frac{1}{8}\left[\int dx + \cancel{\int \cos 2x\,dx} - \tfrac{1}{2}\int dx - \tfrac{1}{2}\int \cos 4x\,dx - \cancel{\int \cos 2x\,dx} + \int \sin^2 2x \cos 2x\,dx\right]$$

$$= \frac{1}{8}\left[\tfrac{1}{2}\int dx - \tfrac{1}{2}\int \cos 4x\,dx + \int \sin^2 2x\cos 2x\,dx\right]$$

$u = 4x$
$du = 4dx$
$\frac{1}{4}du = dx$

$u = \sin 2x \quad du = 2\cos 2x\,dx$
$\frac{1}{2}du = \cos 2x\,dx$

$$= \frac{1}{8}\left[\frac{x}{2} - \frac{1}{2}\cdot\frac{1}{4}\sin 4x + \frac{1}{2}\int u^2\,du\right]$$

$$= \boxed{\frac{1}{8}\left[\frac{x}{2} - \frac{1}{8}\sin 4x + \frac{1}{2}\cdot\frac{1}{3}\sin^3 2x\right] + C}$$

Type 3

1) $\sin mx \cos nx = \frac{1}{2}\left[\sin(m+n)x + \sin(m-n)x\right]$

2) $\sin mx \sin nx = \frac{1}{2}\left[\cos(m+n)x - \cos(m-n)x\right]$

3) $\cos mx \cos nx = \frac{1}{2}\left[\cos(m+n)x + \cos(m-n)x\right]$

These come up a lot when solving PDE's (partial differential equations), but luckily, we usually don't need the above formulas.

Products of even and odd functions (Not in your text)

even · even = even odd · odd = even

even · odd = odd odd · even = odd

think even = (+) odd = (−) then:

$(+)\cdot(+) = (+)$ $(+)\cdot(-) = (-)$

$(-)\cdot(-) = (+)$ $(-)\cdot(+) = (-)$

Q: why do we care whether the product of two functions is odd or even?

A: When we evaluate integrals over intervals which are symmetric with respect to 0, then:

f odd $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$

f even $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

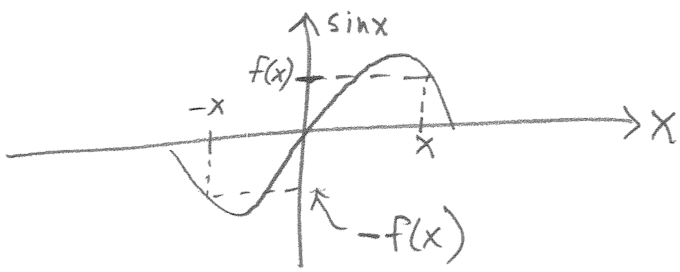
Example

Recall the definition of even and odd functions

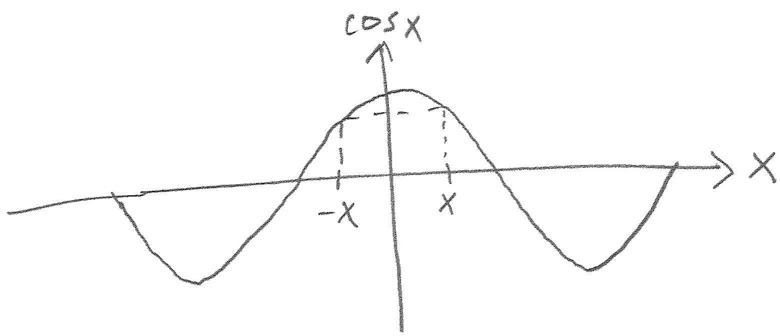
f even $\iff f(-x) = f(x)$

f odd $\iff f(-x) = -f(x)$

Since $\sin(-x) = -\sin x \implies \sin x$ is odd



$\cos(-x) = \cos x \implies \cos x$ is even



$$\int_{-\pi}^{\pi} \underbrace{\sin x \cdot \cos x}_{\substack{\text{odd} \cdot \text{even} \\ \text{odd}}}\, dx = 0$$

scaling input does not affect "oddness" nor "evenness".

$$\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = 0$$

__why?__ Recall that definite integrals correspond to area under the graph. When a function is odd it will have equal amounts of positive and negative area, so they cancel. (Look at the graph of $\sin x$ above to see an example.)

Type (4) $\int \tan^n x\,dx$, $\int \cot^n x\,dx$

Ex. Recall: $\sin^2 x + \cos^2 x = 1$ (Pythagorean identity)

$$\frac{1}{\cos^2 x}\left(\sin^2 x + \cos^2 x = 1\right) \Rightarrow \tan^2 x + 1 = \sec^2 x$$

$$\frac{1}{\sin^2 x}\left(\sin^2 x + \cos^2 x = 1\right) \Rightarrow \boxed{1 + \cot^2 x = \csc^2 x}$$

$$\int \cot^4 x\,dx = \int \cot^2 x\,(\csc^2 x - 1)\,dx$$

$$= \int \cot^2 x \csc^2 x\,dx - \int \cot^2 x\,dx$$

$u = \cot x \quad du = -\csc^2 x\,dx$

$$= -\int u^2\,du - \int (\csc^2 x - 1)\,dx$$

$$= \frac{-1}{3}u^3 + \cot x + x + C$$

$$= \boxed{\frac{-1}{3}\cot^3 x + \cot x + x + C}$$

Type 5 $\int \tan^m x \sec^n x \, dx$, $\int \cot^m x \csc^n x \, dx$

Ex.

$$\int \tan^3 \sec x \, dx = \int \tan^2 x \cdot \tan x \sec x \, dx$$

$$= \int (\sec^2 x - 1) \tan x \sec x \, dx$$

$u = \sec x \qquad du = \sec x \tan x \, dx$

$$= \int (u^2 - 1) \, du$$

$$= \frac{1}{3} u^3 - u + C$$

$$= \boxed{\frac{1}{3} \sec^3 x - \sec x + C}$$