§ 7.2 Integration by Parts

Recall the product rule, where $u = u(x)$, $v = v(x)$:

$$\boxed{D_x[u \cdot v] = u'v + uv'}$$ product rule

$$D_x[u \cdot v] = v\,du + u\,dv$$

$$\int D_x[u \cdot v]\,dx = \int v\,du + \int u\,dv$$

$$u \cdot v = \int v\,du + \int u\,dv$$

$$\Rightarrow \boxed{\int u\,dv = uv - \int v\,du}$$ Memorize this!

Notice there is a dv on the left and only v's on the right hand side of the equation.

Key Idea: This means: If we can break up an integrand into two factors, u and dv, and we can integrate dv to get v, then we can rewrite the difficult integral on the left using the "simpler" rewrite rule on the right.

Similar to how u-substitution "undoes" the chain rule of differentiation, one can think of integration by parts as "undoing" the product rule of differentiation, however this is only an analogy.

Ex. Find $\int x \cos x \, dx$

$$\boxed{\begin{array}{ll} u = x & dv = \cos x \, dx \\ du = dx & v = \sin x \end{array}}$$

When doing integration by parts, I recommend writing both substitutions; i.e. u & dv in the top quadrants of a box.

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx$$

$$\int u \, dv = u \, v - \int v \, du$$

$$= x \sin x - (-\cos x) + C$$

$$= \boxed{x \sin x + \cos x + C}$$

Ex. $\int \underbrace{\ln x}_{u} \underbrace{dx}_{dv}$

$$\boxed{\begin{array}{ll} u = \ln x & dv = dx \\ du = \frac{1}{x} dx & v = x \end{array}}$$

we can finally integrate $\ln x$!

$$\int \ln x \, dx = (\ln x) \cdot x - \int x \cdot \frac{1}{x} dx$$

$$= x \ln x - \int dx$$

$$= \boxed{x \ln x - x + C}$$

Ex. $\int \arcsin x \, dx$

$$\boxed{\begin{array}{ll} u = \arcsin x & dv = dx \\ du = \frac{1}{\sqrt{1-x^2}} dx & v = x \end{array}}$$

Now u-substitute!

$u = 1 - x^2 \quad du = -2x \, dx$

$-\frac{1}{2} du = x \, dx$

$$\int \underbrace{\arcsin x}_{u} \underbrace{dx}_{dv} = x \cdot \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx$$

$$= x \cdot \sin^{-1} x - \frac{-1}{2} \int u^{-1/2} du$$

$$= x \cdot \sin^{-1} x + \cancel{\frac{1}{2}} \cdot \frac{\cancel{2}}{1} u^{1/2} + C$$

$$\boxed{= x \cdot \sin^{-1} x + \sqrt{1-x^2} + C}$$

How to choose u & dv : $\boxed{\text{LIPET}}$

Hard to $\int$

Logs

Inverse Trig.

Polynomials

Exponentials

Trig.

Easy to $\int$

Guideline for how to choose u, whatever is leftover becomes dv.

Memorize This!

Ex. $\int 2xe^{3x}\,dx$

$$\boxed{\begin{array}{ll} u = 2x & dv = e^{3x} \\ du = 2dx & v = \frac{1}{3}e^{3x} \end{array}}$$

$$= \frac{2}{3}xe^{3x} - \frac{2}{3}\int e^{3x}\,dx$$

$$= \frac{2}{3}xe^{3x} - \frac{2}{3}\cdot\frac{1}{3}e^{3x} + C$$

$$= \boxed{\frac{2}{3}xe^{3x} - \frac{2}{9}e^{3x} + C}$$

Ex. $\int e^x \cos x\,dx$

$$\boxed{\begin{array}{ll} u = e^x & dv = \cos x\,dx \\ du = e^x dx & v = \sin x \end{array}}$$

$$\boxed{\begin{array}{ll} u = e^x & dv = \sin x\,dx \\ du = e^x dx & v = -\cos x \end{array}}$$

$$= e^x \sin x - \int e^x \sin x\,dx$$

$$= e^x \sin x - \left[-e^x\cos x + \int e^x \cos x\,dx\right]$$

$$\Rightarrow \int e^x\cos x\,dx = e^x \sin x + e^x\cos x - \int e^x\cos x\,dx \Rightarrow 2\int e^x \cos x\,dx = e^x\sin x + e^x\cos x$$

$$\Rightarrow \int e^x \cos x\,dx = \boxed{\frac{1}{2}e^x\left[\sin x + \cos x\right] + C}$$

Tabular Method: A fast way to integrate (poly.)$\begin{cases}\text{exp.}\\ \text{trig.}\end{cases}$

To find $\int \overset{u}{p(x)} \cdot \overset{dv}{E(x)\,dx}$, write 3 columns:

polynomial

trigonometric or exponential function
(e.g. $\sin x$, or e^{2x})

du derivatives	$\int dv$ integrals	sign(+/−)
Consecutive derivatives of p(x) go here.	Consecutive integrals of E(x) go here.	− + − + − ⋮

first row is always:

P(x)	E(x)	−

Ex. $\int \underbrace{(2x^2-3)}_{P(x)} \underbrace{e^x}_{E(x)}\,dx = (2x^2-3)e^x - 4xe^x + 4e^x + C$

du	$\int dv$	(+/−)
$2x^2-3$	e^x	−
$4x$	e^x	+
4	e^x	−
0	e^x	+

Ex Another Tabular Method Example.

$$\int x^3 \sin(x/2)\,dx = -2x^3\cos(x/2) + 12x^2\sin(x/2) + 48x\cos(x/2) - 96\sin(x/2) + C$$

du	$\int dv$	$(+/-)$
x^3	$\sin(x/2)$	$-$
$3x^2$	$-2\cos(x/2)$	$+$
$6x$	$-4\sin(x/2)$	$-$
6	$8\cos(x/2)$	$+$
0	$16\sin(x/2)$	$-$

You can always check your work in this column by starting at the last entry and differentiating until you hopefully get the first entry.

Hint: This is a great type of test question!

6

34.)

$$\int_0^1 \underbrace{t\,\overbrace{(t-1)^{12}}^{u}\,dt}_{dV} = (t-1)^{12}\cdot\frac{t^2}{2}\Big|_0^1 - \underbrace{\int_0^1 \frac{t^2}{2}\,12(t-1)^{11}\,dt}_{\text{Ugh! This is worse.}}$$

$u = (t-1)^{12}$	$dV = t\,dt$
$du = 12(t-1)^{11}dt$	$V = \frac{t^2}{2}$

X

$u = t$	$dV = (t-1)^{12}$
$du = dt$	$V = \frac{1}{13}(t-1)^{13}$

Try again!

$$= \cancel{\frac{t}{13}(t-1)^{13}\Big|_0^1}^{\,0} - \int_0^1 \frac{1}{13}(t-1)^{13}dt$$

$$= -\frac{1}{13}\cdot\frac{1}{14}(t-1)^{14}\Big|_0^1$$

$$= \frac{-1}{13\cdot 14}\left(0 - (-1)^{14}\right)$$

$$= \frac{1}{13\cdot 14} = \frac{1}{182}$$

The above demonstrates:

$$\int_a^b u\cdot dV = uv\Big|_a^b - \int_a^b v\cdot du$$

that is, the method of integration of parts works for definite integrals too!