

## § 7.2 Integration by Parts

①

Recall the product rule, where  $u = u(x)$ ,  $v = v(x)$ :

$$\boxed{D_x[u \cdot v] = u'v + uv'} \quad \underline{\text{product rule}}$$

$$D_x[u \cdot v] = v du + u dv$$

$$\int D_x[u \cdot v] dx = \int v du + \int u dv$$

$$u \cdot v = \int v du + \int u dv$$

$$\Rightarrow \boxed{\int u dv = uv - \int v du} \quad \underline{\text{Memorize this!}}$$

Notice there is a  $dv$  on the left and only  $v$ 's on the right hand side of the equation.

Key Idea { This means: If we can break up an integrand into two factors,  $u$  and  $dv$ , and we can integrate  $dv$  to get  $v$ , then we can rewrite the difficult integral on the left using the "simpler" rewrite rule on the right.

Similar to how  $u$ -substitution "undoes" the chain rule of differentiation, one can think of integration by parts as "undoing" the product rule of differentiation, however this is only an analogy.

Ex. Find  $\int x \cos x \, dx$

$u = x$	$dv = \cos x \, dx$
$du = dx$	$v = \sin x$

When doing integration by parts, I recommend writing both substitutions, i.e.  $u$  &  $dv$  in the top quadrants of a box.

$$\begin{aligned} \int \underbrace{x}_u \underbrace{\cos x \, dx}_{dv} &= \underbrace{x \sin x}_u \underbrace{- \int \sin x \, dx}_v \\ &= x \sin x - (-\cos x) + C \\ &= \boxed{x \sin x + \cos x + C} \end{aligned}$$

Ex.  $\int \frac{\ln x \, dx}{u \, dv}$

$u = \ln x$	$dv = dx$
$du = \frac{1}{x} \, dx$	$v = x$

We can finally integrate  $\ln x$ !

$$\begin{aligned} \int \ln x \, dx &= (\ln x) \cdot x - \int x \cdot \frac{1}{x} \, dx \\ &= x \ln x - \int dx \\ &= \boxed{x \ln x - x + C} \end{aligned}$$

Ex.  $\int \arcsin x \, dx$

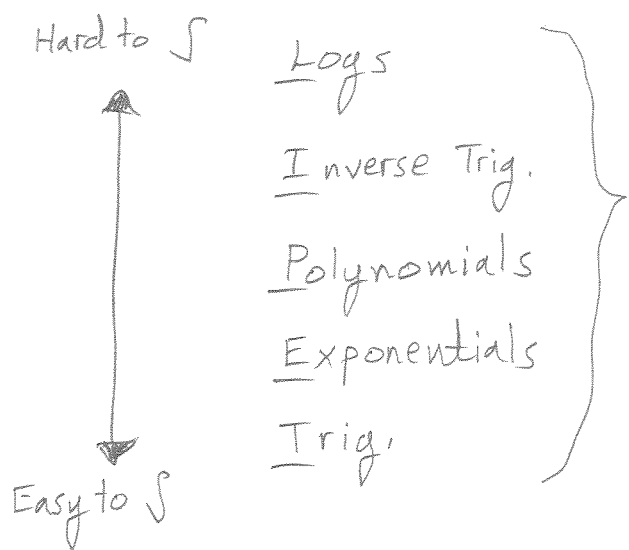
$u = \arcsin x$	$dv = dx$
$du = \frac{1}{\sqrt{1-x^2}} \, dx$	$v = x$

Now  $u$ -substitute!

$$\begin{aligned} \int \underbrace{\arcsin x}_u \underbrace{dx}_{dv} &= x \cdot \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx \\ &= x \cdot \sin^{-1} x - \frac{1}{2} \int u^{-1/2} \, du \\ &= \boxed{x \cdot \sin^{-1} x + \frac{1}{2} \cdot \frac{2}{1} u^{1/2} + C} \\ &= \boxed{x \cdot \sin^{-1} x + \sqrt{1-x^2} + C} \end{aligned}$$

$$\begin{aligned} u &= 1-x^2 \quad du = -2x \, dx \\ \frac{1}{2} du &= -x \, dx \end{aligned}$$

How to choose  $u$  &  $dv$ : LIPET



Guideline for how to choose  $u$ , whatever is leftover becomes  $dv$ .

Memorize This!

Ex.  $\int 2x e^{3x} dx$

$$= \frac{2}{3} x e^{3x} - \frac{2}{3} \int e^{3x} dx$$

$$= \frac{2}{3} x e^{3x} - \frac{2}{3} \cdot \frac{1}{3} e^{3x} + C$$

$$= \frac{2}{3} x e^{3x} - \frac{2}{9} e^{3x} + C$$

$$u = 2x \quad dv = e^{3x}$$

$$du = 2dx \quad v = \frac{1}{3} e^{3x}$$

Ex.  $\int e^x \cos x dx$

$$= e^x \sin x - \int e^x \sin x dx$$

$$= e^x \sin x - \left[ -e^x \cos x + \int e^x \cos x dx \right]$$

$$u = e^x \quad dv = \cos x dx$$

$$du = e^x dx \quad v = \sin x$$

$$u = e^x \quad dv = \sin x dx$$

$$du = e^x dx \quad v = -\cos x$$

$$\Rightarrow \int e^x \cos x dx = e^x \sin x + e^x \cos x - \int e^x \cos x dx \Rightarrow 2 \int e^x \cos x dx = e^x \sin x + e^x \cos x$$

$$\Rightarrow \int e^x \cos x dx = \left[ \frac{1}{2} e^x [\sin x + \cos x] + C \right]$$

Tabular Method: A fast way to integrate (poly.) { exp, trig.

To find  $\int \overset{u}{P(x)} \cdot \overset{dv}{E(x)} dx$ , write 3 columns:  
 polynomial (under  $P(x)$ )  
 trigonometric or exponential function (e.g.  $\sin x$ , or  $e^{2x}$ ) (under  $E(x)$ )

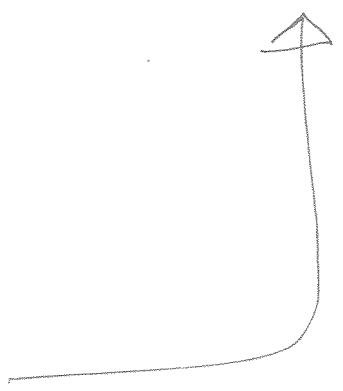
$\frac{du}{dx}$ derivatives	$\int dv$ integrals	sign(+/-)
consecutive derivatives of $P(x)$ go here.	Consecutive integrals of $E(x)$ go here.	-
		+
		-
		+
		-
		⋮

first row is always:  

$P(x)$	$E(x)$	-
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Ex.  $\int \underbrace{(2x^2-3)}_{P(x)} \cdot \underbrace{e^x}_{E(x)} dx = (2x^2-3)e^x - 4xe^x + 4e^x + C$

$du$	$\int dv$	(+/-)
$2x^2-3$	$e^x$	-
$4x$	$e^x$	+
$4$	$e^x$	-
$0$	$e^x$	+



Ex Another Tabular Method Example.

(5)

$$\int x^3 \sin(x/2) dx = -2x^3 \cos(x/2) + 12x^2 \sin(x/2) + 48x \cos(x/2) - 96 \sin(x/2) + C$$

$du$	$\int dv$	(+/-)
$x^3$	$\sin(x/2)$	-
$3x^2$	$-2 \cos(x/2)$	+
$6x$	$-4 \sin(x/2)$	-
$6$	$-8 \cos(x/2)$	+
$0$	$-16 \sin(x/2)$	-

↑ You can always check your work in this column by starting at the last entry and differentiating until you hopefully get the first entry.

Hint: This is a great type of test question!

34.)

$$\int_0^1 \underbrace{t(t-1)^{12}}_{dv} dt = (t-1)^{12} \cdot \frac{t^2}{2} \Big|_0^1 - \underbrace{\int_0^1 \frac{t^2}{2} 12(t-1)^{11} dt}_{\text{Ugh! This is worse}}$$

$u = (t-1)^{12}$	$dv = t dt$
$du = 12(t-1)^{11} dt$	$v = \frac{t^2}{2}$

X

$u = t$	$dv = (t-1)^{12}$
$du = dt$	$v = \frac{1}{13}(t-1)^{13}$

Try again!

$$\begin{aligned} &= \frac{t}{13} (t-1)^{13} \Big|_0^1 - \int_0^1 \frac{1}{13} (t-1)^{13} dt \\ &= -\frac{1}{13} \cdot \frac{1}{14} (t-1)^{14} \Big|_0^1 \\ &= \frac{-1}{13 \cdot 14} (0 - (-1)^{14}) \\ &= \frac{1}{13 \cdot 14} = \frac{1}{182} \end{aligned}$$

The above demonstrates:

$$\int_a^b u \cdot dv = uv \Big|_a^b - \int_a^b v \cdot du$$

that is, the method of integration of parts works for definite integrals too!