§6.9 Hyperbolic Functions and Their Inverses

Def

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

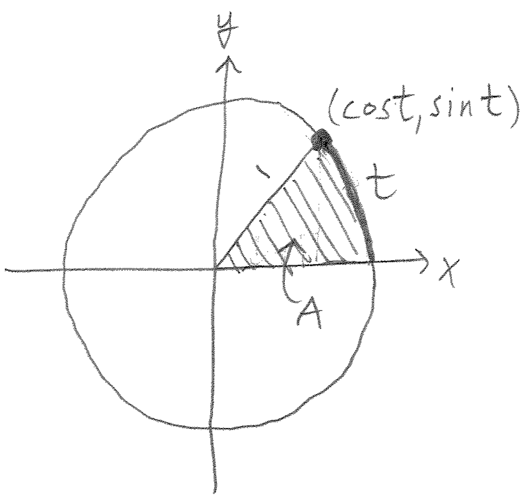
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

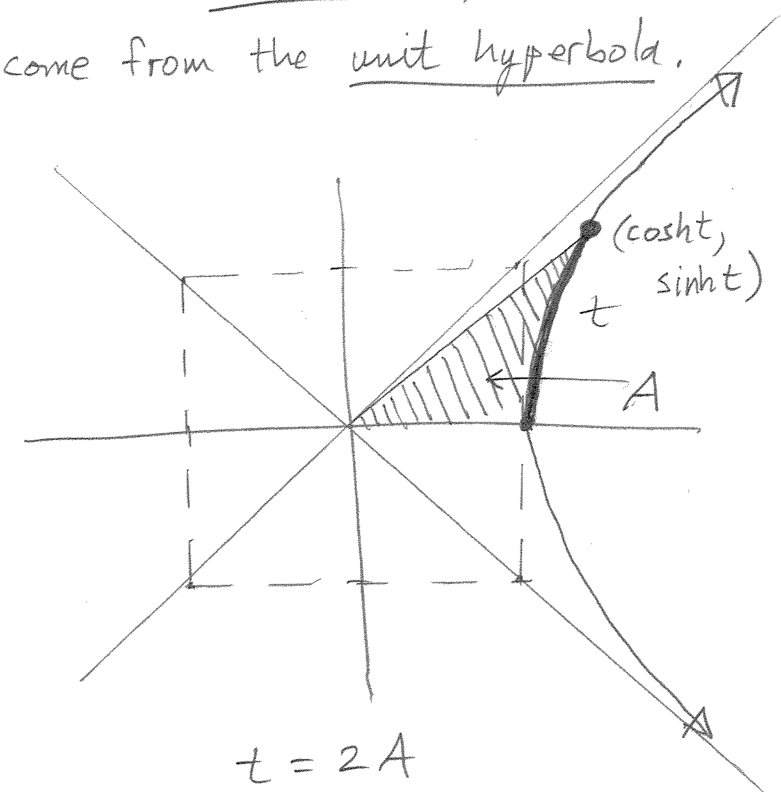
Main Hyperbolic identity: $\boxed{\cosh^2 x - \sinh^2 x = 1}$

notice the negative sign.

The trig. functions are related to the unit circle, but the hyperbolic trig. functions come from the unit hyperbola.



$t = 2A$



$t = 2A$

$$D_x \sinh x = D_x\left[\frac{e^x - e^{-x}}{2}\right] = \frac{e^x - e^{-x}(-1)}{2}$$

$$= \frac{e^x + e^{-x}}{2} = \cosh x.$$

So $\boxed{D_x \sinh x = \cosh x}$

$$D_x \cosh x = D_x \frac{e^x + e^{-x}}{2}$$

$$= \frac{e^x + e^{-x}\cdot(-1)}{2}$$

$$= \frac{e^x - e^{-x}}{2} = \sinh x \implies \boxed{D_x \cosh x = \sinh x}$$

Derivatives

$D_x \sinh x = \cosh x$ | $D_x \cosh x = \sinh x$

$D_x \tanh x = \operatorname{sech}^2 x$ | $D_x \coth x = -\operatorname{csch}^2 x$

$D_x \operatorname{sech} x = - \operatorname{sech} \tanh x$ | $D_x \operatorname{csch} x = -\operatorname{csch} x \coth x$

Notice the 2 differences with the regular trig. functions, (other than everything ending in an "h".)

Ex. $\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} dx$ let $u = \cosh x$ $du = \sinh x \, dx$

$$= \int \frac{1}{u} du = \ln|u| + C = \ln|\cosh x| + C$$

$$= \ln(\cosh x) + C$$

(because $\cosh x > 0$ for all x.)

Inverse Hyperbolic Functions

$x = \sinh^{-1} y \iff \sinh x = y$

$x = \cosh^{-1} y \iff \cosh x = y$

$x = \tanh^{-1} y \iff \tanh x = y$

Q: Can we find an explicit formula for $\sinh^{-1} x$? A: Yes!

$$y = \frac{e^x - e^{-x}}{2} = \sinh x$$

Multiply both sides by $2e^x$,

$$\Rightarrow 2ye^x = \cancel{2}e^x\left[\frac{e^x - e^{-x}}{\cancel{2}}\right]$$

$$= e^{2x} - e^0$$

$$= (e^x)^2 - 1 \quad \Rightarrow \boxed{(e^x)^2 - 2y(e^x) - 1 = 0}$$

$a = 1$, b, c

This is a quadratic equation in disguise, but instead of our variable being just "x", it is "e^x". Use the quadratic formula: $\boxed{ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

$$e^x = \frac{2y \pm \sqrt{4y^2 + 4}}{2} = \frac{\cancel{2}y}{\cancel{2}} \pm \frac{\cancel{2}\sqrt{y^2+1}}{\cancel{2}} = y \pm \sqrt{y^2+1}$$

$$\Rightarrow \boxed{e^x = y \pm \sqrt{y^2+1}}$$

Now solve for x,

$$\ln(e^x) = \ln\left(y \pm \sqrt{y^2+1}\right)$$

(Note: $y - \sqrt{y^2+1} < 0$ and domain of $\ln$ is $(0, \infty)$.)

$$x = \sinh^{-1} y = \ln\left(y + \sqrt{y^2+1}\right)$$

$$\Rightarrow \boxed{\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)}$$

$$D_x \sinh^{-1} x = D_x \ln\left(x + \sqrt{x^2+1}\right)$$

$$= \frac{1}{x + \sqrt{x^2+1}}\left(1 + \tfrac{1}{2}(x^2+1)^{-1/2} \cdot (2x)\right)$$

$$= \frac{1}{x + \sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x + \sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}\right) + \frac{x}{x\sqrt{x^2+1} + x^2+1}$$

$$= \frac{x + \sqrt{x^2+1}}{x\sqrt{x^2+1} + x^2+1}$$

$$= \frac{\cancel{\left(x + \sqrt{x^2+1}\right)}}{\left(x + \sqrt{x^2+1}\right)\sqrt{x^2+1}}$$

$$= \frac{1}{\sqrt{x^2+1}}$$

Connection with differential equations:

- Both $\sinh x$ and $\cosh x$ are solutions to the following, linear, 2^{nd} order, ordinary, differential equation:

$$y'' = y \quad \text{OR} \quad \frac{d^2y}{dx^2} = y$$

Check: Let $y = \sinh x$ $\qquad y = \cosh x$

$$y' = \cosh x \qquad y' = \sinh x$$

$$y'' = \sinh x \checkmark \qquad y'' = \cosh x \checkmark$$

- When (if) you study ODEs (ordinary differential eqns) you will see that the general solution to $y'' = y$ is a linear superposition or linear combination of these two solutions. That is the general solution is:

$$\boxed{y(x) = A\cosh x + B\sinh x}$$ where A, B are constants.

- Similarly, $\cos x$ and $\sin x$ are solutions to

$$y'' = -y \quad \text{OR} \quad \frac{d^2y}{dx^2} = -y$$

Check: $y = \cos x$ $\qquad y = \sin x$

$$y' = -\sin x \qquad y' = \cos x$$

$$y'' = -\cos x \checkmark \qquad y'' = -\sin x \checkmark$$

General solution: $\boxed{y(x) = A\cos x + B\sin x}$.