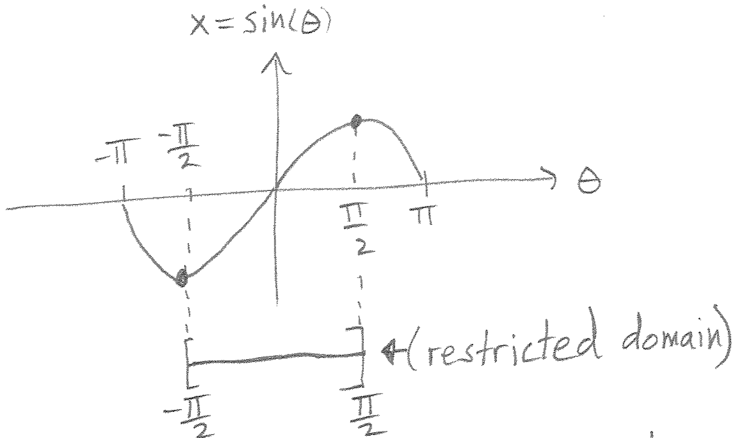
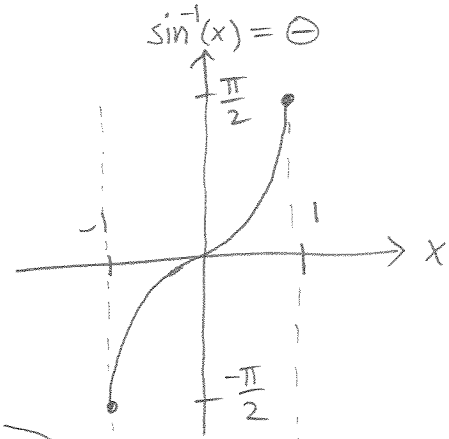
§ 6.8 The Inverse Trigonometric Functions and Their Derivatives

Recall Thm A of section 6.2:

> **Thm A**: If f is strictly monotonic on its domain, then f has an inverse.

The trig functions are horribly non-monotonic! By definition they undulate. However, if we restrict their domains, then we get portions which are strictly monotonic and thus invertible.



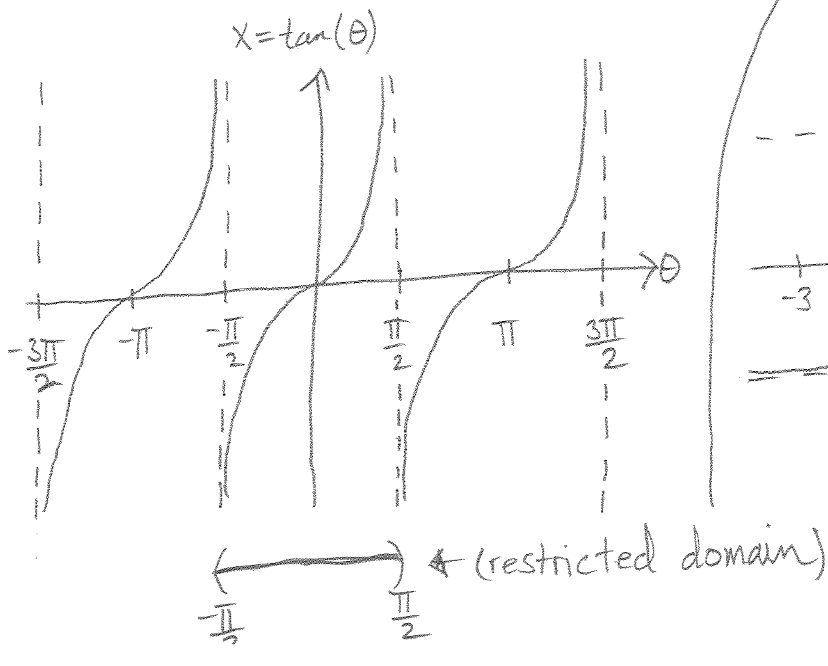


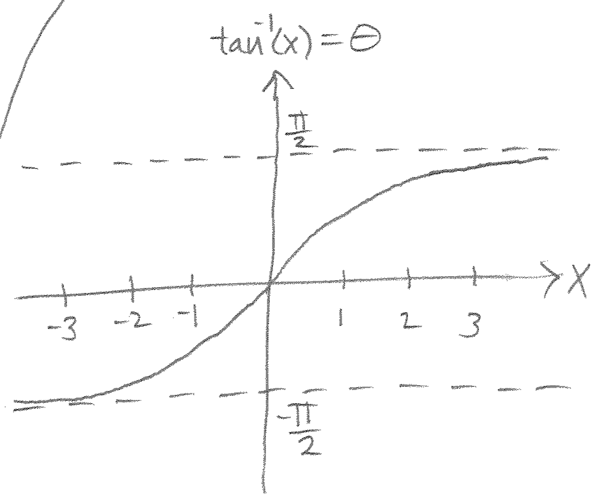
So,

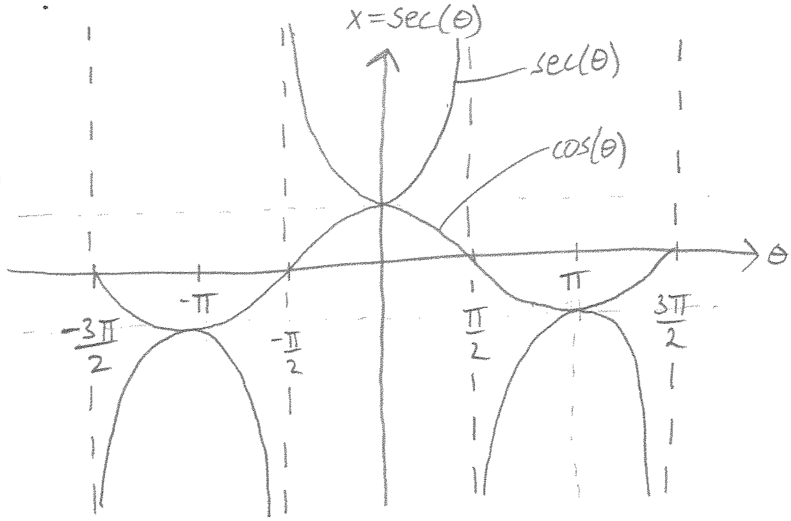
$$x = \sin\theta \iff \sin^{-1}(x) = \theta \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$$

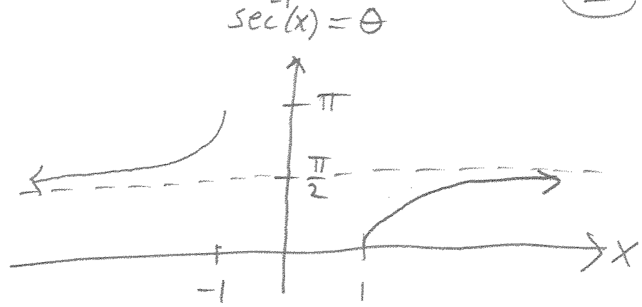
$$x = \cos\theta \iff \cos^{-1}(x) = \theta \qquad (0 \le \theta \le \pi)$$

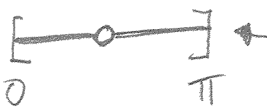
$$x = \tan\theta \iff \tan^{-1}(x) = \theta \qquad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$$











(restricted domain

$$x = \sec(\theta) \iff \sec^{-1}(x) = \theta \qquad (0 \le \theta \le \pi) \text{ AND } \left(\theta \ne \frac{\pi}{2}\right)$$

Recall: $\sec\theta = \dfrac{1}{\cos\theta}$

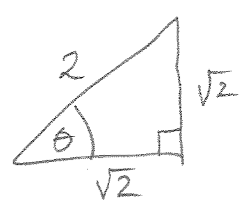
$\csc\theta = \dfrac{1}{\sin\theta}$

$\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

Examples

① Compute: $\sin^{-1}(\sqrt{2}/2)$.

$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \theta \iff \frac{\sqrt{2}}{2} = \sin\theta \iff$



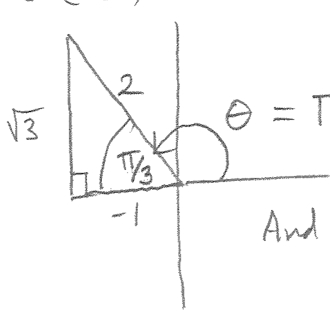
$\iff \boxed{\theta = \frac{\pi}{4}}$

Check if θ is in the restricted domain of sin:

$\frac{\pi}{4} = \theta \in [-\pi/2, \pi/2]$ ✓

② Compute: $\cos^{-1}\left(-\frac{1}{2}\right)$

$\cos^{-1}\left(\frac{-1}{2}\right) = \theta \iff -\frac{1}{2} = \cos(\theta)$

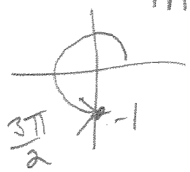


$\theta = \pi - \frac{\pi}{3} = \boxed{\frac{2\pi}{3}}$

And yes $\frac{2\pi}{3} \in [0, \pi]$ ✓

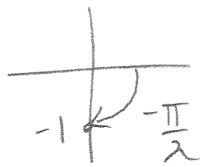
3. Compute: $\cos(\cos^{-1} 0.6) = 0.6$ because they're inverses

4. $\underbrace{\sin^{-1}\left(\sin \frac{3\pi}{2}\right) \neq \frac{3\pi}{2}}_{\text{tricky}}$ because $\frac{3\pi}{2} \notin [-\pi/2, \pi/2]$



$\Rightarrow \sin^{-1}\left(\sin \frac{3\pi}{2}\right) = \sin^{-1}(-1) = \theta \iff -1 = \sin\theta$

$\Rightarrow$



$\Rightarrow \boxed{\theta = \frac{-\pi}{2}} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ ✓

so $\sin^{-1}\left(\sin \frac{3\pi}{2}\right) = \frac{-\pi}{2}$

$x = \sec\theta = \frac{1}{\cos\theta} \Rightarrow \frac{1}{x} = \cos\theta$

$\Rightarrow \cos^{-1}\left(\frac{1}{x}\right) = \theta$

$\qquad\qquad\qquad\qquad \|$

But, $x = \sec\theta \iff \sec^{-1}(x) = \theta$

$\boxed{\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right) = \theta}$

$x = \csc\theta = \frac{1}{\sin\theta} \Rightarrow \frac{1}{x} = \sin\theta$

$\Rightarrow \sin^{-1}\left(\frac{1}{x}\right) = \theta$

$\qquad\qquad\qquad\qquad \|$

But, $x = \csc\theta \iff \csc^{-1}(x) = \theta$

$\boxed{\csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right) = \theta}$

$x = \cot\theta = \frac{1}{\tan\theta} \Rightarrow \frac{1}{x} = \tan\theta$

$\Rightarrow \tan^{-1}\left(\frac{1}{x}\right) = \theta$

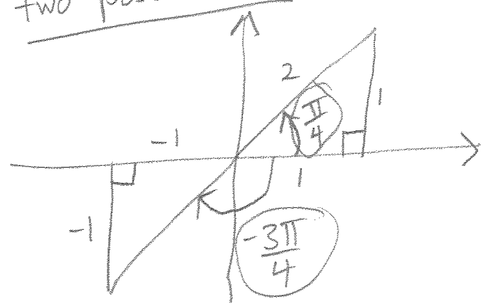
$\qquad\qquad\qquad\qquad \|$

But, $x = \cot\theta \iff \cot^{-1}(x) = \theta$

$\boxed{\cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right) = \theta}$

Examples

two possibilities:

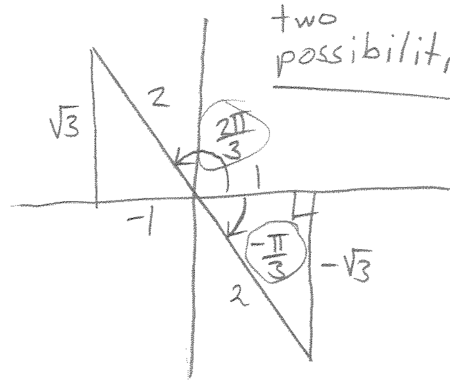


a) $\theta = \tan^{-1}(1) \iff \tan\theta = 1$

but $-\frac{3\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ = restricted domain of $\tan\theta$

$\implies \theta = \frac{\pi}{4}$ so $\boxed{\tan^{-1}(1) = \frac{\pi}{4}}$.

two possibilities



b.) $\theta = \tan^{-1}(-\sqrt{3}) \iff \tan(\theta) = -\sqrt{3}$

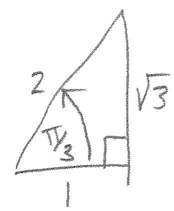
but $\frac{2\pi}{3} \notin \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ = restricted domain

$\implies \theta = \frac{-\pi}{3}$ so $\boxed{\tan^{-1}(-\sqrt{3}) = \frac{-\pi}{3}}$.

c) $\sec^{-1}(-1) = \cos^{-1}\left(\frac{1}{-1}\right) = \cos^{-1}(-1) = \pi$

see prev. page

d) $\sec^{-1}(2) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

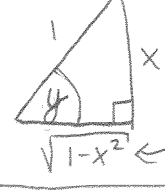


Thm $\boxed{D_x \sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \quad -1<x<1}$

pf Let $y = \sin^{-1}x$ thus $\sin y = x$. Now differentiate both sides with respect to x: $D_x[\sin y] = D_x[x] \implies \cos y \cdot D_x y = 1$ chain rule

$\implies \cos(\sin^{-1}x) \cdot D_x y = 1$

$\implies D_x y = \frac{1}{\cos(\sin^{-1}x)}$



$y = \sin^{-1}x \implies \cos(\sin^{-1}x) = \sqrt{1-x^2}$

by Pythagorean Theorem

$\implies D_x[\sin^{-1}x] = \frac{1}{\sqrt{1-x^2}}$. $\square$

Thm

$$D_x \tan^{-1} x = \frac{1}{1+x^2}$$

pf Let $\boxed{y = \tan^{-1} x}$ so $\tan y = x$. Now differentiate both sides with respect to x,

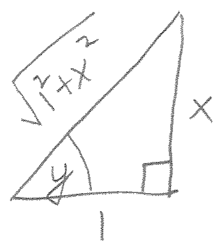
chain rule

$$D_x[\tan y] = D_x x \implies \sec^2 y \cdot D_x y = 1 \implies$$

$$D_x y = \frac{1}{\sec^2 y} \implies D_x \tan^{-1} x = \frac{1}{\sec^2 y}$$

but $\tan y = x$

(recall $\tan = \frac{\text{opp}}{\text{adj.}}$)



$$\implies \sec y = \frac{1}{\cos y} = \frac{1}{\frac{1}{\sqrt{1+x^2}}}$$

$$\implies \sec^2 y = 1 + x^2$$

$$\implies D_x \tan^{-1} x = \frac{1}{1+x^2}. \quad \square$$

The same process as in the above two examples yields this table:

1) $D_x \sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}$ $\quad -1 < x < 1$

2) $D_x \cos^{-1} x = \dfrac{-1}{\sqrt{1-x^2}}$ $\quad -1 < x < 1$

3) $D_x \tan^{-1} x = \dfrac{1}{1+x^2}$

4) $D_x \sec^{-1} x = \dfrac{1}{|x|\sqrt{x^2-1}}$ $\quad |x| > 1$ equivalently $x > 1$ OR $x < -1$.

Ex Find $D_x \sin^{-1}(\overbrace{3x-1}^{u(x)}) = \dfrac{1}{\sqrt{1-u^2}} \cdot D_x u$

$$= \frac{1}{\sqrt{1-(3x-1)^2}} \cdot 3$$

$$= \frac{3}{\sqrt{1-(9x^2-6x+1)}} = \frac{3}{\sqrt{6x-9x^2}}$$

Ex, $D_x \sin^{-1}(x) = \dfrac{1}{\sqrt{1-x^2}} \iff \displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$

$D_x \sin^{-1}(u) = \dfrac{1}{\sqrt{1-u^2}} D_x u \iff \displaystyle\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}u + C$

But we can go further: $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx = ?$

Notice: $\sqrt{4-x^2} = \sqrt{4\left(1+\frac{x^2}{4}\right)} = 2\sqrt{1-\left(\frac{x}{2}\right)^2}$ thus

$\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx$ let $u = \frac{x}{2}$ $\quad du = \frac{1}{2}dx$

$2du = dx$

$$= \cancel{\tfrac{1}{2}} \int \frac{\cancel{2}}{\sqrt{1-u^2}}\,du$$

$$= \boxed{\sin^{-1}\left(\frac{x}{2}\right) + C}$$

$\sqrt{a^2-x^2} = \sqrt{a^2\left(1-\frac{x^2}{a^2}\right)} = a\sqrt{1-\left(\frac{x}{a}\right)^2}$ $\quad u = \frac{x}{a} \quad du = \frac{1}{a}dx$

$a\,du = dx$

$\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{a\sqrt{1-(x/a)^2}}\,dx = \int \frac{\cancel{a}\,du}{\cancel{a}\sqrt{1-u^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C.$ $\rightarrow$

$$\boxed{\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right)+C}$$

But instead of remembering a formula, it is better to remember and use the process by which it was obtained!

Ex Evaluate $\displaystyle\int \frac{1}{\sqrt{12-9x^2}}\,dx$

$$\sqrt{12-9x^2} = \sqrt{12\left(1-\frac{9x^2}{12}\right)} = \sqrt{12}\sqrt{\left(1-\frac{3x^2}{4}\right)}$$

$$=\sqrt{12}\sqrt{\left(1-\left(\frac{\sqrt{3}x}{2}\right)^2\right)} \qquad \left(\sqrt{12}=\sqrt{3\cdot 2^2} = 2\sqrt{3}\right)$$

$$\int \frac{1}{\sqrt{12-9x^2}}\,dx = \frac{1}{2\sqrt{3}}\int \frac{1}{\sqrt{1-\left(\frac{\sqrt{3}x}{2}\right)^2}}\,dx \qquad \text{let } u=\frac{\sqrt{3}x}{2}$$

$$du = \frac{\sqrt{3}}{2}\,dx$$

$$= \frac{1}{2\sqrt{3}}\cdot\frac{2}{\sqrt{3}}\int\frac{1}{\sqrt{1-u^2}}\,du \qquad \frac{2}{\sqrt{3}}\,du = dx$$

$$= \frac{1}{3}\sin^{-1}(u)+C = \boxed{\frac{1}{3}\sin^{-1}\left(\frac{\sqrt{3}x}{2}\right)+C}$$

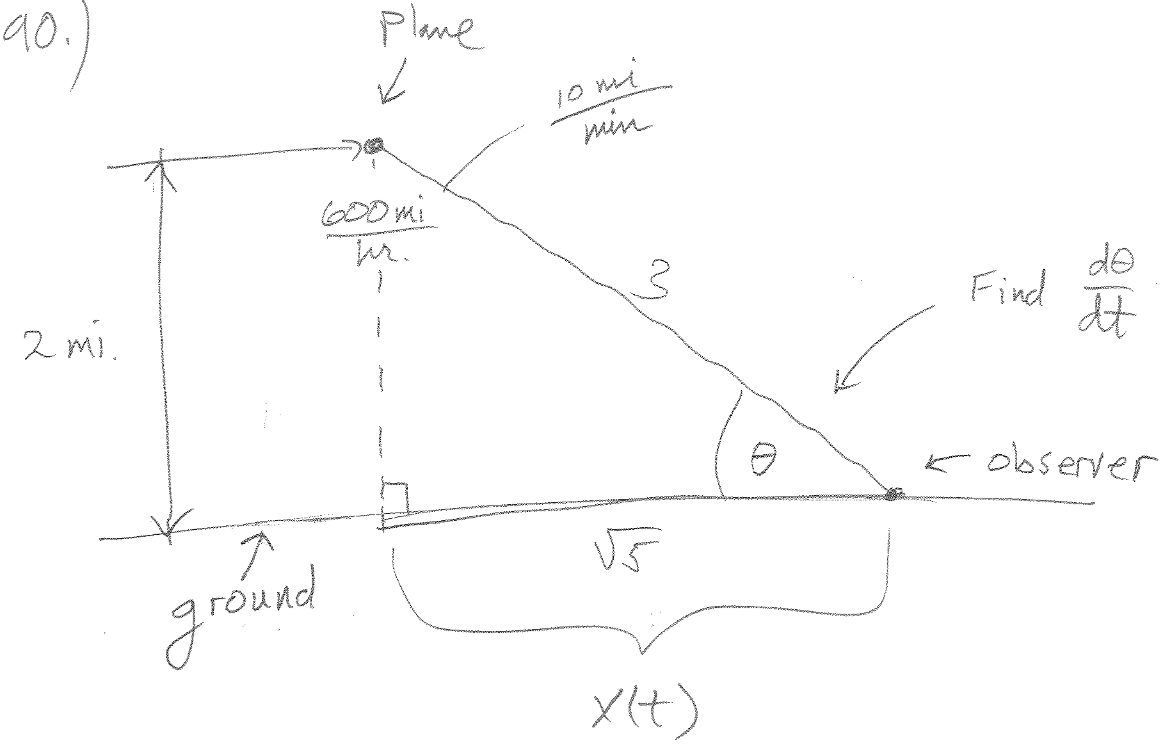
Ex. Completing the Square

Evaluate : $\displaystyle\int\frac{7}{x^2-6x+25}\,dx$ $\qquad x^2-6x = x^2-6x+9-9 = (x-3)^2-9$

$$= 7\int\frac{1}{(x-3)^2-9+25}\,dx = 7\int\frac{1}{(x-3)^2+16}\,dx = 7\int\frac{1}{(x-3)^2+4^2}\,dx$$

$$\boxed{= \frac{7}{4}\tan^{-1}(x-3)+C}$$

90.)



$$x(t) = \sqrt{5} - 10t$$

$$\theta(t) = \tan^{-1}\left(\frac{2}{x(t)}\right) = \tan^{-1}\left(\frac{2}{\sqrt{5} - 10t}\right)$$

$$\frac{d\theta}{dt} = \frac{1}{1 + \left(\frac{2}{\sqrt{5} - 10t}\right)^2} \cdot \frac{-2}{(\sqrt{5} - 10t)^2}(-10)$$

$$\frac{d\theta}{dt} = \frac{20}{(\sqrt{5} - 10t)^2 + 4}$$

When $t = 0$ $\quad \frac{d\theta}{dt} = \frac{20}{9} \approx 2.22 \ \frac{\text{rad}}{\text{min.}}$