§ 6.6 First Order, Linear Differential Equations

The following equation can NOT be solved via our method of the previous section (separation of variables).

$$\frac{dy}{dx} = 2x - 3y \quad \text{OR} \quad y' = 2x - 3y$$

When this fails we say the equation is <u>not separable</u>.

However, if the differential equation can be rearranged to the form:

$$\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)}$$

then we can use the integrating factor method.

⓪ Put into above form

① compute the <u>integrating factor</u> $I = e^{\int P(x)\,dx}$

(Here we can safely ignore the +C when computing $\int P(x)\,dx$.)

② Multiply both sides of the diff. eq. by the integrating factor $I = e^{\int P(x)\,dx}$

③ Recognize the left-hand-side as the derivative of a product, and integrate both sides.

<u>Ex.</u> $\frac{dy}{dx} = \frac{\sin 3x}{x^2} - \frac{2}{x}y \xRightarrow{⓪} \frac{dy}{dx} + \overset{P(x)}{\boxed{\frac{2}{X}}}\,y = \overset{Q(x)}{\boxed{\frac{\sin 3x}{x^2}}}$

① $I = e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln|x|}$ (No +C !)

$= e^{\ln|x|^2} = |x|^2 = \boxed{x^2}$ this is our integrating factor!

② $$\left[\frac{dy}{dx} + \frac{2}{x}y\right] x^2 = \frac{\sin 3x}{x^2} \cdot x^2$$

$$\frac{dy}{dx} \cdot x^2 + 2yx = \sin 3x$$

$$\boxed{y' \cdot x^2 + 2yx} = \sin 3x$$

Notice:

$$D_x[y \cdot x^2] = \boxed{y' \cdot x^2 + 2yx}$$

③ $$D_x[y \cdot x^2] = \sin 3x$$

$$\int D_x[y \cdot x^2]\,dx = \int \sin 3x\,dx$$

$u = 3x$

$du = 3\,dx$

$\frac{1}{3}du = dx$

$$= \frac{1}{3}\int \sin u\,du$$

1st FTC ↓

$$y \cdot x^2 = \frac{-1}{3}\cos(3x) + C$$

(Now we need the "+c".)

$$y(x) = \frac{1}{x^2}\left[\frac{-1}{3}\cos(3x) + C\right]$$

$$\boxed{y(x) = \frac{-1}{3x^2}\cos(3x) + \frac{C}{x^2}}$$

General Solution

The general solution is a family of solutions, where each solution in the family corresponds with a single value of C.

A particular solution is a single solution from the family of solutions that satisfies an initial condition such as $y = a$ when $x = b$. It is a point that lies on the graph of y.

Ex.

⓪ $$\frac{dy}{dx}\overset{P(x)}{-3}y = \overset{Q(x)}{xe^{3x}}$$ and satisfies initial condition: $y=4$ when $x=0$

① $$I = e^{\int P(x)\,dx} = e^{\int -3\,dx} = e^{-3x}$$ (No "+c" here!)

② $$\left[\frac{dy}{dx} - 3y\right]e^{-3x} = xe^{3x}\cdot e^{-3x}$$

$$y'e^{-3x} - 3ye^{-3x} = x$$

Notice: $D_x\left[ye^{-3x}\right] = y'e^{-3x} - 3ye^{-3x}$

③ $$D_x\left[ye^{-3x}\right] = x$$

$$\int D_x\left[ye^{-3x}\right]dx = \int x\,dx$$

$$ye^{-3x} = \frac{x^2}{2} + c$$ ("+c" here!)

$$\boxed{y(x) = \frac{x^2}{2}e^{3x} + ce^{3x}}$$ General Solution

Use I.C. to solve for c:

$$4 = \frac{0^2}{2}\cdot e^0 + c\cdot e^0$$

$$4 = 0 + c\cdot 1 \implies \boxed{c=4}$$

$$\boxed{y(x) = \frac{x^2}{2}e^{3x} + 4e^{3x}}$$ Particular Solution

Ex. A tank initially contains 200 gallons of brine, holding 50 lbs, of salt in solution. Salt water containing 2 lbs. salt per gallon is entering the tank at the rate of 4 gallons per minute. The mixture is kept uniform by constant stirring, and the mixture flows out of the tank at a rate of 4 gallons per minute. Find the amount of salt in the tank after 40 min.

Solution

Let y = # lbs. of salt in the tank

$$\frac{dy}{dt} = \text{rate in} - \text{rate out}$$

c_i = concentration in

c_o = " out

f_i = flow rate in

f_o = " " out

$$\frac{dy}{dt} = c_i f_i - c_o f_o$$

$$\frac{dy}{dt} = \frac{2\text{ lb.}}{\cancel{\text{gal.}}} \cdot \frac{4\,\cancel{\text{gal.}}}{\text{min.}} - \frac{y\text{ lb.}}{200\,\cancel{\text{gal.}}} \cdot \frac{4\,\cancel{\text{gal.}}}{\text{min}}$$

$$\frac{dy}{dt} = 8\,\frac{\text{lb.}}{\text{min}} - \frac{y}{50}\,\frac{\text{lb.}}{\text{min}}$$

$$\frac{dy}{dt} + \overset{P(t)}{\left(\frac{1}{50}\right)} y = \overset{Q(t)}{(8)} \qquad \boxed{\text{I.F.} = e^{\int \frac{1}{50} dt} = e^{t/50}}$$

$$\left[y' + \frac{1}{50} y\right] e^{t/50} = 8e^{t/50}$$

$$y' e^{t/50} + \frac{1}{50} y e^{t/50} = 8e^{t/50}$$

$$\int D_t\left[y e^{t/50}\right] dt = \int 8e^{t/50}\, dt$$

$u = t/50 \qquad du = \frac{1}{50}\,dt$

$50\,du = dt$

$$y e^{t/50} = 8 \cdot 50 \int e^u\, du = 400 e^{t/50} + C$$

$\longrightarrow$

Ex. cont.)

$y \cdot e^{t/50} = 400 e^{t/50} + C$

General Solution

$y(t) = 400 + \frac{C}{e^{t/50}}$ OR $\boxed{y(t) = 400 + Ce^{-t/50}}$

Now use the initial condition $y(0) = 50$ (see 1st sentence.), to get a particular solution:

$y(0) = 50 \implies 50 = 400 + Ce^{0}$ $(e^0 = 1)$

$\implies C = 50 - 400$

$\boxed{C = -350}$

Thus $\boxed{y(t) = 400 - 350 e^{-t/50}}$

$y(40) = 400 - 350 e^{-40/50} \approx$ $\boxed{242.74 \text{ lbs. of salt}}$