§6.4 General Exponential and Logarithmic Functions

Q: Without using a calculator how could you compute $e^{\sqrt{2}}$?

A: Recall $e^x = \exp x$ and $x = \exp y \iff y = \ln x$

Thus: $e^{\sqrt{2}} = \exp\sqrt{2} \iff \sqrt{2} = \ln e^{\sqrt{2}}$

Further recall: $\ln x = \int_1^x \frac{1}{t}\,dt$

Thus we can approximate $e^{\sqrt{2}}$ by varying x until:

$$\sqrt{2} = \int_1^x \frac{1}{t}\,dt,$$

but this requires many tedious Riemann sums. (In section 9.9 we will learn a much easier technique.)

Problem Up til this point we have only been able to compute powers such as a^r when both a and r are rational, now we'll use the exponential function to extend this to things like $2^{\sqrt{2}}$, π^π, π^e, etc.

Def For $a>0$, $\boxed{a^x = e^{x\ln a}}$

Why? $a^x = \exp(\ln a^x) = \exp(x \cdot \ln a) = e^{x \ln a}$

Ex. $3^2 = e^{2\ln 3} = e^{2(1.0986123)} = 9.0000$ (Try this on your calculator.)

Properties of Exponents (Thm A)

$a>0,\ b>0 \quad x, y \in \mathbb{R}$

① $a^x a^y = a^{x+y}$ ④ $(ab)^x = a^x b^x$

② $\dfrac{a^x}{a^y} = a^{x-y}$ ⑤ $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

③ $(a^x)^y = a^{xy}$

pf ① $a^x a^y = e^{\ln(a^x a^y)} = e^{(\ln a^x + \ln a^y)} = e^{(x\ln a + y \ln a)}$

$= e^{(x+y)\ln a} = e^{\ln(a^{(x+y)})} = a^{x+y}$ $\boxed{1}$

④ $(ab)^x = e^{\ln((ab)^x)} = e^{x \ln ab} = e^{x(\ln a + \ln b)}$

$= e^{(x\ln a + x \ln b)} = e^{(\ln a^x + \ln b^x)} \overset{①}{=} e^{\ln a^x} e^{\ln b^x} = a^x b^x$ $\boxed{4}$

② and ③ are proved in the book, ⑤ is similar to ④.

Differentiation and Integration of Exponential Functions (Thm B)

$$\boxed{\begin{aligned} &D_x a^x = a^x \ln a \\ &\int a^x\,dx = \left(\frac{1}{\ln a}\right) a^x + C \qquad a \neq 1 \end{aligned}}$$

pf $D_x a^x \underset{\substack{\uparrow \\ \text{def of } a^x}}{=} D_x e^{x\ln a} = e^{x \ln a} \cdot D_x[x \ln a] = (\ln a) e^{x \ln a} = (\ln a) a^x$

$= a^x \ln a.$

So $D_x a^x = a^x \ln a \Rightarrow \dfrac{1}{\ln a} D_x a^x = a^x$

$\Rightarrow \displaystyle\int \left(\frac{1}{\ln a} D_x a^x\right) dx = \int a^x dx \Rightarrow \int a^x dx = \frac{1}{\ln a} a^x + C.$ $\square$

$$\boxed{\begin{aligned} D_x\, a^u &= a^u \cdot \ln a \cdot u' \\ \int a^u\,du &= \left(\frac{1}{\ln a}\right)\cdot a^u + C \end{aligned}}$$

Ex. Find $D_x\, 3^{\sqrt{x}}$: let $u = \sqrt{x}$ $\quad a = 3 \quad u' = \frac{1}{2}x^{-\frac{1}{2}}$

$$= \frac{1}{2\sqrt{x}}$$

$$\Rightarrow D_x\, 3^{\sqrt{x}} = 3^{\sqrt{x}} \cdot \ln 3 \cdot \frac{1}{2\sqrt{x}}$$

$$\boxed{= \frac{3^{\sqrt{x}} \cdot \ln 3}{2\sqrt{x}}}$$

Ex. $y = (x^4+2)^5 + 5^{(x^4+2)}$ Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 5(x^4+2)^4 \cdot 4x^3 + 5^{(x^4+2)} \cdot \ln 5 \cdot (4x^3)$$

$$= \boxed{4x^3\left[5(x^4+2)^4 + 5^{(x^4+2)} \cdot \ln 5\right]}.$$

Ex. Find $\int 2^{x^3} \cdot x^2\,dx$ $\quad$ let $u = x^3$ $\quad du = 3x^2\,dx$

$$\frac{1}{3}du = x^2\,dx$$

$$= \frac{1}{3}\int 2^u \cdot du$$

$$= \frac{1}{3} \cdot \frac{1}{\ln 2} \cdot 2^{x^3} + C$$

$$= \boxed{\frac{2^{x^3}}{3\ln 2} + C}$$

The 2 Key Formulas

$$\boxed{\begin{aligned} &(1) \quad a^x = e^{x\ln a} \\ &(2) \quad \log_a x = \frac{\ln x}{\ln a} \end{aligned}}$$

We defined "exponentiation base a" via formula (1) above.
Formula (2) above comes about by defining $\log_a x$ to be the inverse of a^x, "exponentiation base a".

<u>Def</u> $a>0$ $a\neq 1$, then

$$\boxed{y = \log_a x \iff x = a^y}$$

Let's use this definition to get formula (2) above

$$\underline{y=\log_a x} \Rightarrow x = a^y \Rightarrow \ln x = \ln a^y \Rightarrow$$

$$\Rightarrow \ln x = y\ln a$$

$$\Rightarrow \underline{y = \frac{\ln x}{\ln a}}$$

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

<u>Ex</u>

$$\log_3 100 = \frac{\ln 100}{\ln 3}$$

Since $\ln a$ is just a constant we immediately deduce,

- $D_x \log_a x = D_x \dfrac{\ln x}{\ln a} = \dfrac{1}{\ln a} \cdot D_x \ln x = \dfrac{1}{\ln a} \cdot \dfrac{1}{x} = \dfrac{1}{x \ln a}$

$$\boxed{D_x \log_a x = \frac{1}{x \ln a}}$$

And if u is a function of x:

$$\boxed{D_x \log_a u = \frac{u'}{u \ln a}}$$

<u>chain rule version</u>

Note: $\int \ln x \, dx$ and $\int \log_a x \, dx$ will be covered in section 7.2.

<u>Ex.</u> $y = \log_{10}(x^4 + 13)$ $\quad a = 10$ $\quad$ let $u = x^4 + 13$

$u' = 4x^3$

- $$\boxed{y' = \frac{4x^3}{(x^4+13)\cdot \ln 10}}$$

<u>Ex.</u> A new way to compute y' when $y = x^x$.

Recall that we did this before via <u>logarithmic differentiation</u>:

$y = x^x$

$\ln y = \ln x^x = x \ln x$

$\frac{1}{y} \cdot y' = 1 \cdot \ln x + x \cdot \frac{1}{x}$

$= 1 + \ln x$

$y' = x^x(1 + \ln x)$

<u>New way</u>:

$y = x^x = e^{\ln x^x} = e^{x \ln x}$

$y' = e^{x \cdot \ln x} \cdot (1 + \ln x)$

$y' = e^{\ln x^x} \cdot (1 + \ln x)$

$y' = x^x(1 + \ln x).$

Power Function vs. Exponential Function

$$x^a \qquad\qquad a^x$$

$$D_x[x^a] = a x^{a-1} \qquad\qquad D_x[a^x] = a^x \cdot \ln a$$

$$\int x^a \, dx = \frac{1}{a+1} x^{a+1} + C \quad (a \neq -1) \qquad\qquad \int a^x \, dx = \frac{1}{\ln a} \cdot a^x + C$$

Inverse Function

$$y = x^a \iff x = \sqrt[a]{y} = y^{\frac{1}{a}}$$

Inverse Function

$$y = a^x \iff \log_a y = x$$

Ex. $y = (x^2+1)^{\pi} + \pi^{\sin x}$

$$y' = \pi(x^2+1)^{\pi-1} \cdot (2x) + \pi^{\sin x} \cdot \ln\pi \cdot \cos x$$

Ex. $\displaystyle\int_{1/2}^{1} \frac{5^{1/x}}{x^2} \, dx$ let $u = \frac{1}{x} = x^{-1}$

$$du = -x^{-2} \, dx$$

$$du = \frac{-1}{x^2} \, dx$$

$$-du = \frac{dx}{x^2}$$

$$= -\int_{x=1/2}^{x=1} 5^u \, du = \frac{1}{\ln 5} 5^{\frac{1}{x}} \Big|_{1/2}^{1}$$

$$= \frac{-1}{\ln 5}\left[5 - 5^2\right] = \boxed{\frac{20}{\ln 5}}$$