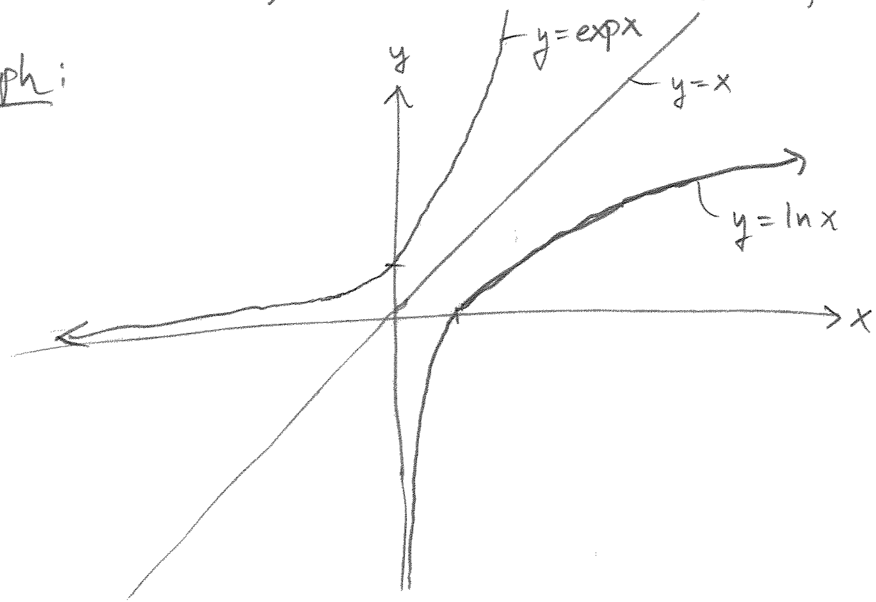
§6.3 The Natural Exponential Function

Def Since we defined $\ln(x) = \int_1^x \frac{1}{t}\,dt$ that is, as an accumulation function of area under the curve $\frac{1}{t}$ starting at 1, $\ln(x)$ is strictly monotonic and thus has an inverse. We will define the inverse by the following equivalence:

$$\boxed{y = \ln x \iff x = \exp y}$$

In other words, we're simply defining $\exp(y)$ to be the inverse of ln, i.e. $\ln^{-1}(x) = \exp(x)$, and $\exp^{-1}(x) = \ln(x)$.

Graph:



Notice that these two graphs are mirror images of each other with respect to the line $y=x$. This is true of all functions and their inverses.

Def Let e denote the unique positive real number such that $\ln e = 1$, i.e. $\int_1^e \frac{1}{t}\,dt = 1$.

$\boxed{e \approx 2.71828\ldots}$ (e is transcendental)

Properties of the exponential function:

① $\exp(\ln x) = x \quad x > 0$

② $\ln(\exp y) = y$ for all y.

Q: why do we call the inverse of the natural logarithm the "exponential function"?

A: $e^r = \exp(\ln e^r) = \exp(r \ln e) = \exp(r \cdot 1) = \exp(r)$

Thus $\boxed{e^r = \exp r}$ Note: we will mostly use the e^r notation instead of $\exp r$.

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① $e^{\ln x} = x \quad x > 0$

② $\ln(e^y) = y$ for all y.

These are exactly the same as ① & ② above, but written in our new notation.

Thm If $a, b \in \mathbb{R}$

Then $\boxed{e^a e^b = e^{a+b}}$ AND $\boxed{\dfrac{e^a}{e^b} = e^{a-b}}$

proof $e^a e^b \overset{①}{=} \exp(\ln e^a e^b)$

$= \exp(\ln e^a + \ln e^b)$ by properties of the logarithm.

$\overset{②}{=} \exp(a + b)$

$= e^{a+b}$. □ (similarly for $\frac{e^a}{e^b} = e^{a-b}$)

The most important property of the exponential func. e^x is that:

$\boxed{D_x e^x = e^x}$ or if $y = e^x \Rightarrow y' = y$

<u>proof</u> Let $y = e^x$ so $x = \ln y$.

Differentiate both sides with respect to x (using the chain rule on the right hand side.)

$1 = \frac{1}{y} \cdot D_x y \Rightarrow D_x y = y \Rightarrow D_x[e^x] = e^x$. $\square$

<u>Ex</u> $y = e^{2x^2 - x}$ find y'

$y' = (e^{2x^2 - x}) \cdot \underbrace{(4x - 1)}_{\text{via chain rule.}}$

In general, if $u = f(x)$ then $\boxed{D_x e^u = e^u D_x[u]}$

<u>Ex.</u> Find $D_x e^{(x^2 \ln x)}$

$u = x^2 \ln x$ so $u' = 2x \ln x + x^2 \cdot \frac{1}{x} = 2x \ln x + x = x(1 + 2\ln x)$

$\Rightarrow D_x e^{x^2 \ln x} = e^u D_x[u]$

recall $2\ln x = \ln x^2$

$= e^{x^2 \ln x} \cdot x(1 + \ln x^2)$

Since $D_x e^x = e^x$ it follows that $\boxed{\int e^x dx = e^x + C}$

In general $\boxed{\int e^u\,du = e^u + C}$

Ex. $\int e^{-4x}\,dx$ let $u = -4x$ $du = -4\,dx$ $-\frac{1}{4}\,du = dx$

$$= \int -\frac{1}{4} e^u\,du = \frac{-1}{4}\int e^u\,du = \frac{-1}{4}\left[e^u + c\right]$$

$$= \boxed{\frac{-1}{4} e^{-4x} + C}$$

Ex. $\int x^2 e^{-3x^3}\,dx$ $u = -3x^3$ $du = -9x^2\,dx$

$= \frac{-1}{9}du$ $\frac{-1}{9}\,du = x^2\,dx$

$$= \int \frac{-1}{9} e^u\,du = \left(\frac{-1}{9} e^u + c\right)$$

$$= \boxed{\frac{-1}{9} e^{-3x^3} + C}$$

Ex. $\int \frac{6e^{1/x}}{x^2}\,dx = \int 6x^{-2} e^{x^{-1}}\,dx$ $u = x^{-1}$ $du = -x^{-2}\,dx$

$-du = x^{-2}\,dx$

$$= -6\int e^u\,du$$

$$= -6e^u + C$$

$$= -6e^{x^{-1}} + C$$

$$= \boxed{-6e^{1/x} + C}$$