§ 6.2 Inverse Functions and Their Derivatives

Recall that a function establishes a relationship between two sets, or even between the elements of a single set. We denote this by:

$$f: X \longrightarrow Y \qquad \text{or} \qquad f: X \longrightarrow X$$

If our set is the set of real numbers which we denote by the symbol $\mathbb{R}$, then we write $f: \mathbb{R} \to \mathbb{R}$.

We know how to "undo" simple functions like $f(x) = x+2$ or $g(x) = 3x$. To "undo" f we simply subtract 2 and to undo g we simply divide by 3.

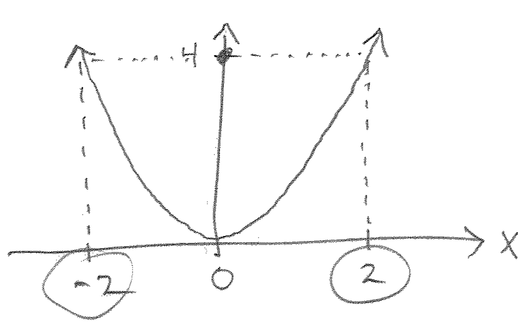
Q: Can we always "undo" a function in one "consistent" way?

A: No. !!

Q: What do you mean by "consistent"?

A: I mean, if I give you the output of a function f, say $f(x) = 4$, then is there one single value which x must be? (We want the "undoing" operation to be a function too.)

Ex $f(x) = x^2$ if $f(x) = x^2 = 4$ then $x = 2$ OR $x = -2$!



"Two inputs correspond with one output."

So we see that in order for a function to be "undone" consistently, the graph of the function must intersect all horizontal lines in just one point. This is known as the <u>horizontal line test.</u> (No wiggles or humps)

If a function can be "undone", we will say it is <u>invertible</u>, and denote the <u>inverse of f</u> by <u>f^{-1}</u>.

$$\boxed{\underline{\text{NOTE:}} \quad f^{-1}(x) \neq \frac{1}{f(x)} \; !}$$

The algebraic operation associated with the inverse of f is <u>function composition</u> NOT multiplication.

<u>function composition:</u> $\boxed{(f \circ g)(x) = f(g(x))}$ "f composed with g is f of g"

If $f: X \to Y$ and f is invertible, i.e. f^{-1} exists, then

$f^{-1}: Y \to X$ <u>and</u> $\boxed{(f^{-1} \circ f)(x) = id(x) \;\; \underline{\text{and}} \;\; (f \circ f^{-1})(y) = id(y)}$

where "id" is short for the <u>identity function.</u>

By definition $id(x) = x$ for all x, it is the function which does absolutely nothing to its input. Another way to write the last box above is: $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$ & $(f \circ f^{-1})(y) = f(f^{-1}(y)) = y$

Def A function is invertible when it is <u>1-1</u> <u>(one to one)</u> meaning

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ "Distinct inputs yield distinct outputs."

Equivalently, (via the contrapositive of the last statement)

f is 1-1 if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

"Equivalent outputs demand equivalent inputs.

For many of the functions we will work with, it is difficult to demonstrate that they are 1-1, it is easier to show monotonicity.

Def f is monotone increasing if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

f is monotone decreasing if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Def f is strictly monotonic if f is monotone increasing or monotone decreasing.

> Thm A If f is strictly monotonic on its domain, then f has an inverse.

In practice we will check for monotonicity via the derivative because monotone increasing corresponds with positive slope, i.e. $f'(x) > 0$ for all x. Similarly for monotone decreasing.

Ex. Any polynomial with only odd degree and positive terms has an inverse. Let $f(x) = x^7 + 2x^3 + x$.

Since $f'(x) = 7x^6 + 6x^2 + 1 > 0$ for all x

$\Rightarrow$ f is strictly monotonic $\xRightarrow{\text{Thm A}}$ f has an inverse.

Q: If a function has an inverse either because it is 1-1, or because it is strictly monotonic (Thm A), can we always find a formula for it?

A: No. ☹

Ex Let $f(x) = x^5 + 2x + 1$

$f'(x) = 5x^4 + 2 > 0$ for all $x \overset{\text{Thm A}}{\Longrightarrow} f^{-1}$ exists!

But sadly we can't solve $y(x) = x^5 + 2x + 1$ for x as a function of y, i.e. $x(y) = ?$ p.s. $x(y) = f^{-1}(y)$.

But sometimes we can! ☺

Ex. $y(x) = 2x + 3 = f(x)$ $f^{-1}(x) = ?$ (or $f^{-1}(y)$ if you prefer)

Solution: $y = 2x + 3$

$\Downarrow$ ① Rewrite with x & y replacing each other

$x = 2y + 3$

$\Downarrow$ ② Solve for y

$$2y = x - 3$$
$$y = \tfrac{1}{2}(x-3) = \boxed{\tfrac{1}{2}x - \tfrac{3}{2} = f^{-1}(x)}$$

Let's check that $f^{-1}(x)$ satisfies: $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

$$f^{-1}(f(x)) = f^{-1}(2x+3) = \tfrac{1}{2}(2x+3) - \tfrac{3}{2} = x \checkmark$$

$$f(f^{-1}(x)) = f\left(\tfrac{1}{2}x - \tfrac{3}{2}\right) = 2\left(\tfrac{1}{2}x - \tfrac{3}{2}\right) + 3 = x \checkmark$$

Thus $f^{-1} \circ f = id$ AND $f \circ f^{-1} = id$.

Example Find $f^{-1}(x)$ when $f(x) = mx + b$. $(m \neq 0)$

Solution

$$\boxed{y = mx + b}$$

$\Downarrow$ ① Rewrite

$$x = my + b$$

$\Downarrow$ ② Solve for y

$$my = x - b$$

$$y = \frac{1}{m}(x - b)$$

$$\boxed{f^{-1}(x) = y = \frac{1}{m}x - \frac{b}{m}}$$

Notice that the slopes are multiplicative inverses of each other!

That is $f'(x) = m$ (for all x) and $(f^{-1})'(x) = \frac{1}{m}$ (for all x)

and $m \cdot \frac{1}{m} = 1$! $\Rightarrow$ $\boxed{(f^{-1})'(x) = \frac{1}{f'(x)}}$

Q: Does this relationship between derivatives hold true for general functions besides simple linear functions?

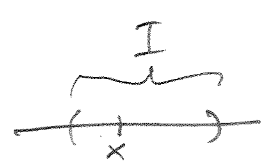
A: YES!

Thm Inverse Function Theorem

If f is ① differentiable (smooth)

② strictly monotonic

③ $f'(x) \neq 0$ for some $x \in I$.



Then

$$\boxed{(f^{-1})'(x) = \frac{1}{f'(x)}}$$

This theorem is extremely important for doing calculus in a non-Euclidean space such as on a torus