§6.1 The Natural Logarithm Function

- $D_x[x^n] = nx^{n-1} \implies D_x[\,?\,] = x^{-1}$

Try: $D_x[x^0] = 0 \cdot x^{-1}$ Nope

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \implies \int x^{-1}\,dx = \,?$

If we apply the usual integration rule we get:

$\int x^{-1}\,dx \overset{?}{=} \frac{1}{-1+1}x^{-1+1} + C = \frac{1}{0}x^0 + C$ ← Uh oh! Division by zero!

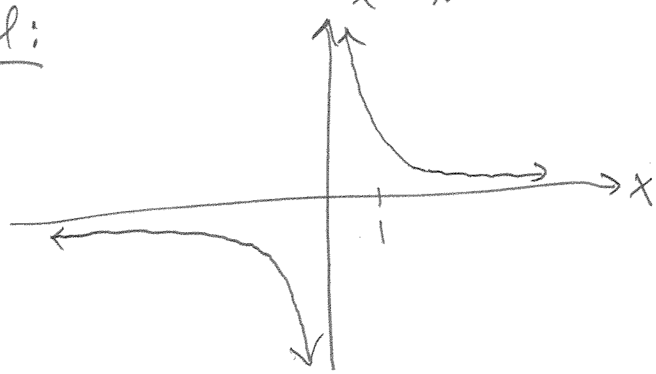
The 1st FTC can help us. Recall that it tells us how to find the derivative of an accumulation function,

- $$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

when f is continuous on $[a,b]$ and $x \in (a,b)$.

Recall:

$f(x) = \frac{1}{x} = x^{-1}$



This is continuous on $(-\infty, 0) \cup (0, \infty)$

Define $\boxed{\ln x = \int_1^x \frac{1}{t}\,dt}$

Note: By defining $\ln x$ as an accumulation function, it is only defined for $x > 0$!

- "ln" or "ℓn" stands for natural logarithm.

* This notation was invented by Irving Stringham, a math professor at Berkeley in 1893.

This definition combined with the 1st FTC gives us:

$$D_x[\ln x] = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x} \quad (= x^{-1})$$

Actually we can show that $D_x[\ln|x|] = \frac{1}{x}$:

There are two cases to consider:

1. $x > 0 \Rightarrow |x| = x$

$$\Rightarrow D_x[\ln|x|] = D_x[\ln x] = \frac{1}{x}$$

2. $x < 0 \Rightarrow |x| = -x$

chain Rule

$$\Rightarrow D_x[\ln|x|] = D_x[\ln(-x)] = \frac{1}{-x} \cdot D_x[-x]$$

$$= \frac{1}{-x} \cdot (-1)$$

$$= \frac{1}{x}$$

Thus:

$$\boxed{\begin{aligned} D_x[\ln|x|] &= \frac{1}{x} \\ \int \frac{1}{x}\,dx &= \ln|x| + C \end{aligned}}$$

Example: Find $\int \frac{3}{3x+5}\,dx$

Use a u-substitution: $u = 3x+5 \qquad du = 3\,dx$

$$\Rightarrow \int \frac{3}{3x+5}\,dx = \int \frac{du}{u} = \ln|u| + C$$

$$= \boxed{\ln|3x+5| + C}$$

This new function opens up a new class of integrals we can compute!

$$\boxed{\int \frac{p(x)}{q(x)}\,dx \quad \text{where } p(x) \text{ \& } q(x) \text{ are polynomials and } \deg p(x) \geq \deg q(x).}$$

Example $\int \frac{x^2}{x-1}\,dx$

(1) First divide!

$$\begin{array}{r} x+1 \quad + \frac{1}{x-1} \\ x-1 \overline{)\, x^2+0x+0} \\ -(x^2-x) \\ \hline x+0 \\ -(x-1) \\ \hline 1 \end{array}$$

(2) $$\int \frac{x^2}{x-1}\,dx = \int \left(x+1+\frac{1}{x-1}\right)dx$$

$$= \int (x+1)\,dx + \int \frac{1}{x-1}\,dx$$

let $u=x-1$
$du=dx$

$$= \boxed{\frac{x^2}{2}+x + \ln|x-1| + C}$$

Trig Integrals (This only works in some situations)

Example $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

def of $\tan x$

$u=\cos x$
$du=-\sin x\,dx \Rightarrow -du=\sin x\,dx$

$$= \int \frac{-1}{u}\,du$$

$$= \boxed{-\ln|\cos x| + C}$$

Logarithmic Differentiation

Facts

1. $\ln 1 = 0$
2. $\ln ab = \ln a + \ln b$
3. $\ln \frac{a}{b} = \ln a - \ln b$
4. $\ln a^r = r \ln a$

We can combine the properties above with implicit differentiation to differentiate complex functions more easily.

Example $y = x^x$ find y'

First take the log of both sides:

$$\ln y = \ln(x^x) \overset{4}{=} x \ln x$$

Now use implicit differentiation:

$$\frac{1}{y} \cdot y' = \overbrace{1 \cdot \ln x + x \cdot \frac{1}{x}}^{\text{product rule}}$$

$$\frac{1}{y} \cdot y' = \ln x + 1$$

$$y' = y(\ln x + 1)$$

$$\boxed{y' = x^x(1 + \ln x)}$$