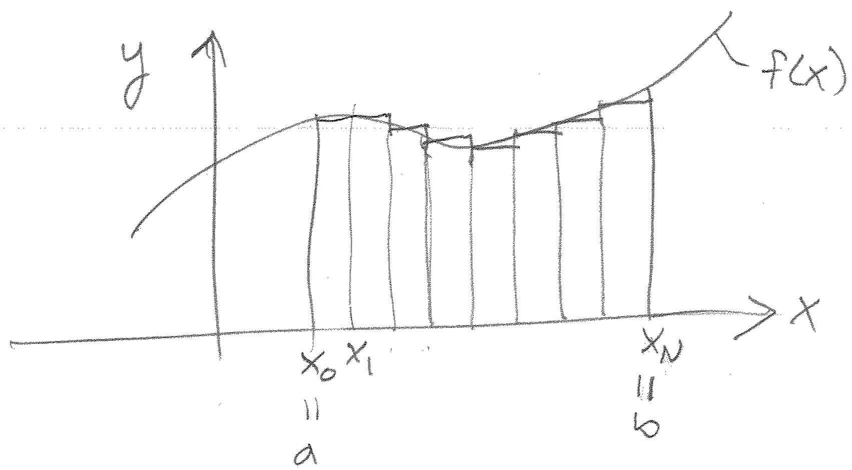
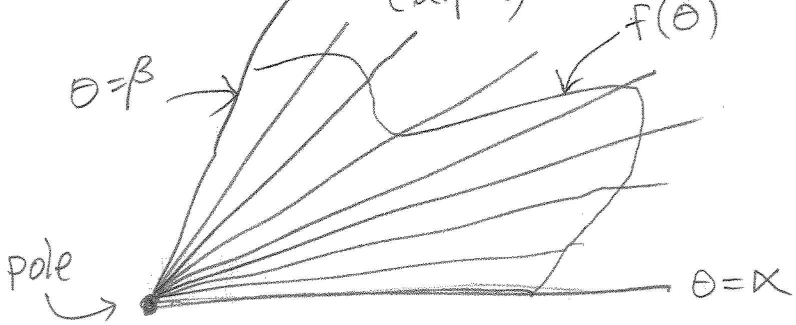
§10.7 Calculus in Polar Coordinates

Recall that our approach to calculation of area under a curve in regular Cartesion coordinates was to partition the x-axis into ever smaller intervals, and then sum the area of all rectangles which lie under the graph and which lie between the limits of integration, usually $[a, b]$.



$$A \approx \sum_{i=0}^{N-1} f(x_i)\Delta x_i \qquad \lim_{N\to\infty} \sum_{i=0}^{N-1} f(x_i)\Delta x_i = \int_a^b f(x)\,dx$$

where $\Delta x_i = (x_{i+1} - x_i)$. In polar coordinates, we don't have rectangles, we have sectors. Recall that polar functions are usually written such that r is a function of θ (theta), i.e. $r = f(\theta)$. So a polar function sweeps out a sector between a starting angle α (alpha) and an end angle β (beta).



- When computing area via polar functions we will follow the same basic strategy of "divide and conquer". That is we will still divide the initial sector into ever smaller sectors and then take the limit of the sum as N, the number of sectors, goes to infinity.

<u>Q</u>: How do we compute the approximate area of a sector?

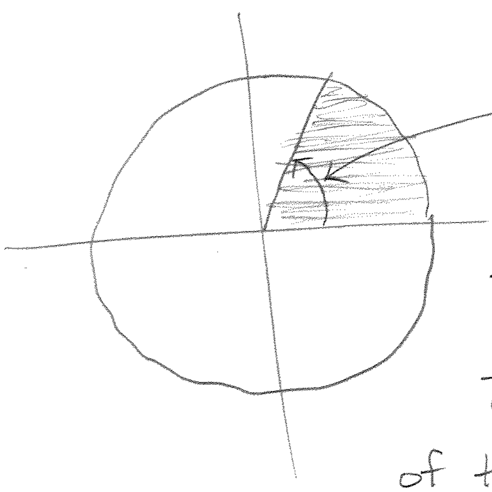
<u>A</u>: We'll use sectors of a circle.

Recall that the area of a circle is $\boxed{A = \pi r^2}$

Thus a sector of that circle with central angle θ radians is:

<u>Area of sector</u> $= \left(\frac{\theta}{2\pi}\right) \cdot \pi r^2$

↑ fraction of total area



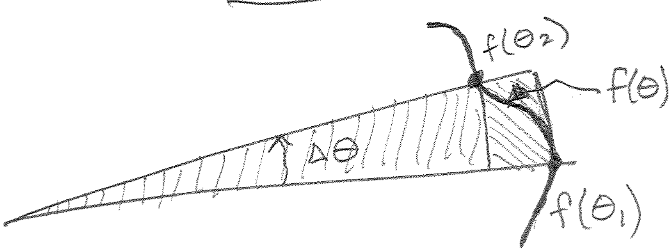
central angle $\theta = \frac{\pi}{3}$ radians

The circle has a total sector of 2π radians

thus $\frac{\theta}{2\pi} = \frac{\pi/3}{2\pi} = \frac{\cancel{\pi}}{3} \cdot \frac{1}{2\cancel{\pi}} = \frac{1}{6}$

Thus a sector of angle $\theta = \frac{\pi}{3}$ "consumes" $\frac{1}{6}$ of the total area of the circle.

In general: $\boxed{\underline{\text{Area of sector}} = \frac{1}{2}\theta r^2}$



Depending on whether we use $r = f(\theta_1)$ or $r = f(\theta_2)$ we will overestimate or underestimate the area!

where

$$A \approx \sum_{i=0}^{N-1} \frac{1}{2}\Delta\theta_i\,[f(\theta)]^2 = \frac{1}{2}\sum_{i=0}^{N-1} [f(\theta)]^2\,\Delta\theta_i$$

$\Delta\theta_i = \theta_{i+1} - \theta_i$

and $\alpha = \theta_0$

$\beta = \theta_N$.

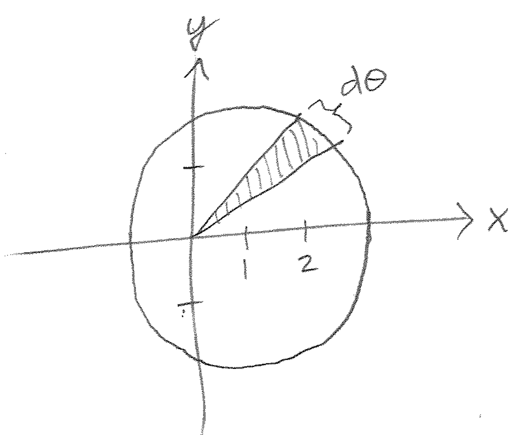
in the limit as $N \to \infty$

$$\boxed{A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\, d\theta}$$

Ex. "Sanity Check" Find the area of the unit circle which we know to be $\pi \cdot 1^2 = \pi$, and where $f(\theta) = 1$.

$$A = \frac{1}{2}\int_0^{2\pi} 1^2 \cdot d\theta = \frac{1}{2}\,\theta\Big|_0^{2\pi} = \frac{1}{2}(2\pi - 0) = \pi \checkmark$$

Ex. Find the area of the region inside the limaçon $\boxed{r = 2 + \cos\theta}$



Because $\cos(-\theta) = \cos(\theta)$, the function above is even, and thus has graph which is symmetric with respect to the x-axis. So we can double the area we get from $0 \le \theta \le \pi$.

$$A = 2 \cdot \frac{1}{2}\int_0^{\pi} (2 + \cos\theta)^2\, d\theta = \int_0^{\pi} (4 + 4\cos\theta + \cos^2\theta)\, d\theta$$

Recall:

$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$

$$= \int_0^{\pi} 4\,d\theta + 4\int_0^{\pi} \cos\theta\, d\theta + \frac{1}{2}\int_0^{\pi} (1 + \cos(2\theta))\, d\theta$$

$$= 4\theta\Big|_0^{\pi} + \cancelto{0}{4\sin\theta\Big|_0^{\pi}} + \frac{\theta}{2}\Big|_0^{\pi} + \cancelto{0}{\frac{1}{4}\sin(2\theta)\Big|_0^{\pi}}$$

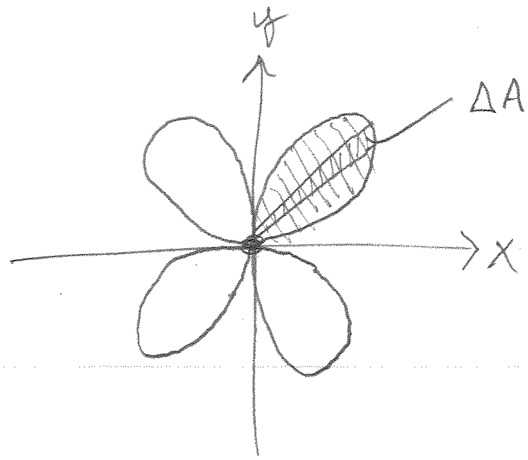
$$= 4\pi + \frac{\pi}{2} = \boxed{\frac{9\pi}{2}} \approx 14.1 > 12.6 \approx \pi \cdot 2^2$$

Area of circle of radius 2

<u>Ex</u> Find the <u>area of one leaf</u> of the four-leaved rose,

$$\boxed{r = 4\sin 2\theta.}$$

<u>Solution</u>



$$\Delta A \approx \frac{1}{2}\left[f(\theta)\right]^2 \Delta\theta \rightsquigarrow \boxed{A = \frac{1}{2}\int_{\alpha}^{\beta}\left[f(\theta)\right]^2 d\theta}$$

Double Angle Formula:

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$A = \frac{1}{2}\int_0^{\pi/2}\left[4\sin 2\theta\right]^2 d\theta = \frac{1}{2}\int_0^{\pi/2} 16\sin^2(2\theta)\,d\theta$$

$$= 8\int_0^{\pi/2}\frac{1}{2}(1-\cos(4\theta))\,d\theta$$

$u = 4\theta \qquad du = 4\,d\theta$

$\frac{1}{4}du = d\theta$

$$= 4\left[\int_0^{\pi/2} 1\,d\theta - \int_0^{\pi/2}\cos 4\theta\,d\theta\right]$$

$$= 4\left[\theta\Big|_0^{\pi/2} - \frac{1}{4}\sin 4\theta\Big|_0^{\pi/2}\right]$$

$$= 4\left[\left(\frac{\pi}{2}-0\right) - \frac{1}{4}\left(\cancelto{0}{\sin 2\pi} - \cancelto{0}{\sin 0}\right)\right]$$

$$= \boxed{2\pi}$$