Math 1220 Exam 1 February 3, 2012

Name: Solutions

Instructions:

- Answer the questions in the space provided.
- You must show your work in order to get credit! Writing just an answer is worth 0 points, even if the answer is correct.
- Partial credit will be awarded.
- The instructor has extra scratch paper if you need it.
- Graphing and scientific calculators are allowed, but smartphones and computers are not allowed.
- This exam is closed book and closed notes, except you may use one double sided 8.5 by 11 inch page of notes.

1. [10 points] Find y' if $y = \ln\left(\frac{2x}{3x+4}\right)$.
(Hint: Use the properties of the logarithm to simplify before differentiating.)

$$y = \ln(2x) - \ln(3x+4)$$

$$y' = \frac{1}{2x} \cdot 2 - \frac{1}{3x+4} \cdot 3$$

$$\boxed{y' = \frac{1}{x} - \frac{3}{3x+4}}$$

2. [10 points] Let $y = \dfrac{\sqrt{x^2+1}}{(9x-4)^2}$. Use logarithmic differentiation to find y'.

$$\ln y = \ln\left[\frac{\sqrt{x^2+1}}{(9x-4)^2}\right]$$

$$\ln y = \ln (x^2+1)^{1/2} - \ln(9x-4)^2$$

$$\ln y = \tfrac{1}{2}\ln(x^2+1) - 2\ln(9x-4)$$

$$\frac{1}{y}\cdot y' = \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x - 2\,\frac{1}{9x-4}\cdot 9$$

$$\frac{1}{y}\cdot y' = \frac{x}{x^2+1} - \frac{18}{9x-4}$$

$$\boxed{y' = \frac{\sqrt{x^2+1}}{(9x-4)^2}\left[\frac{x}{x^2+1} - \frac{18}{9x-4}\right].}$$

3. [10 points] Find the inverse function of $f(x) = 2(x-1)^2 - 1, \quad x > 1.$

$y = 2(x-1)^2 - 1$

$x = 2(y-1)^2 - 1$

$x+1 = 2(y-1)^2$

$\frac{x+1}{2} = (y-1)^2$

$\sqrt{\frac{x+1}{2}} = y-1 \Rightarrow y = \sqrt{\frac{x+1}{2}} + 1$

$\Rightarrow \boxed{f^{-1}(x) = \sqrt{\frac{x+1}{2}} + 1}$

Check:

$f^{-1}(f(x)) = f^{-1}(2(x-1)^2 - 1)$

$= \sqrt{\frac{[2(x-1)^2-1]+1}{2}} + 1$

$= \sqrt{(x-1)^2} + 1 = x$ ✓

$f(f^{-1}(x)) = f\left(\sqrt{\frac{x+1}{2}} + 1\right)$

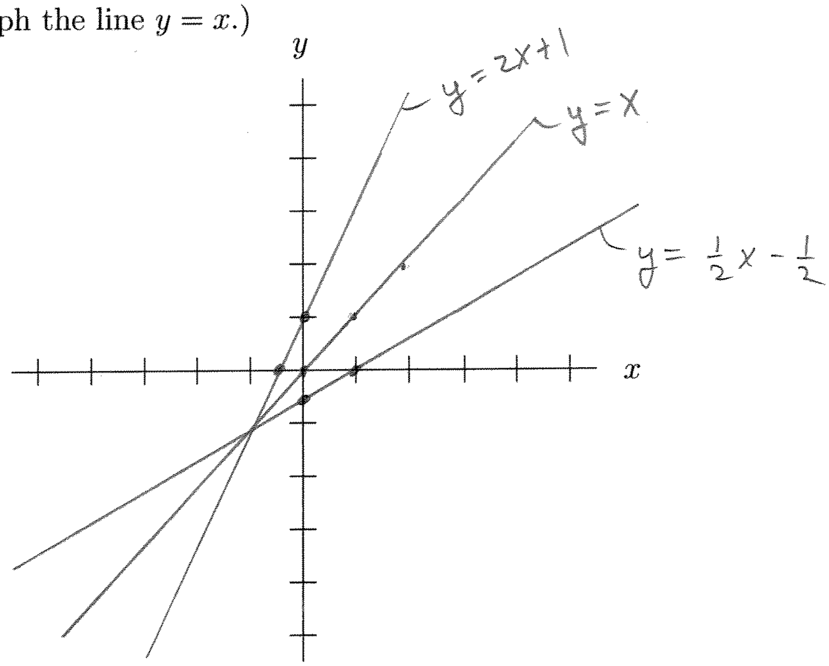
$= 2\left[\left(\sqrt{\frac{x+1}{2}} + 1\right) - 1\right]^2 - 1$

$= 2\left(\sqrt{\frac{x+1}{2}}\right)^2 - 1 = 2 \cdot \frac{x+1}{2} - 1$

$= x + 1 - 1$

$= x$ ✓

4. [10 points] Graph both the function $y = 2x + 1$ and its inverse on the same coordinate axes. (Hint: Also graph the line $y = x$.)



$y = 2x + 1$

$x = 2y + 1$

$x - 1 = 2y$

$\boxed{y = \frac{x-1}{2} = \frac{1}{2}x - \frac{1}{2}}$

5. [10 points] Evaluate the expression: $\ln\left(\frac{1}{\sqrt[3]{e}}\right)$. $= \ln e^{-\frac{1}{3}} = -\frac{1}{3}\ln e^{1}$

$$\boxed{= \frac{-1}{3}}$$

6. [10 points] Find $\int \frac{e^{4x}}{1+e^{4x}}\,dx$.

$u = 1+e^{4x}$ $\quad du = 4e^{4x}\,dx$

$\frac{1}{4}du = e^{4x}\,dx$

$$\int \frac{e^{4x}}{1+e^{4x}}\,dx = \frac{1}{4}\int \frac{1}{u}\,du$$

$$= \frac{1}{4}\ln|u| + C$$

$$= \frac{1}{4}\ln|1+e^{4x}| + C = \boxed{\frac{1}{4}\ln(1+e^{4x}) + C}$$

7. [10 points] Evaluate $\int_0^3 2^{x^2} x\,dx$.

$u = x^2$ $\quad du = 2x\,dx$ $\quad \frac{1}{2}du = x\,dx$

$$\int_0^3 2^{x^2} x\,dx = \frac{1}{2}\int_{x=0}^{x=3} 2^u\,du = \frac{1}{2}\,\frac{1}{\ln 2}\cdot 2^{x^2}\Big|_0^3$$

$$= \frac{1}{2\ln 2}\left[2^9 - 2^0\right]$$

$$= \frac{511}{2\ln 2}$$

8. [10 points] Solve the population equation below, that is, find $P(t)$. Show all steps of the solution.

$$\frac{dP}{dt} = kP$$

pts

Just writing $P(t) = P_0 e^{kt}$

2

5 $\frac{dP}{P} = k\,dt$

6 $\int \frac{dP}{P} = \int k\,dt$

7 $\ln|P| = kt + C$

8 $e^{\ln|P|} = e^{(kt+C)}$

$|P| = e^{kt} \cdot e^{C}$

9 $P(t) = e^{kt} \cdot e^{C}$

Let $P_0 = P(0)$ then $P(0) = e^{0} \cdot e^{C} = P_0$

thus $e^{C} = P_0$, so

10 $P(t) = P_0 e^{kt}$

9. [10 points] A bone is found to contain 30% of the carbon-14 that it contained when it was part of a living organism. How long ago did the organism die?
(The half-life of carbon-14 is 5730 years.)

First find k:

$$\frac{1}{2}\cancel{P_0} = \cancel{P_0}e^{k\cdot 5730}$$

$$\ln\left(\tfrac{1}{2}\right) = \ln\left(e^{k\cdot 5730}\right) = k\cdot 5730$$

$$\Rightarrow k = \frac{\ln\left(\frac{1}{2}\right)}{5730} \approx -1.209068 \times 10^{-4}$$

$$.3\cancel{P_0} = \cancel{P_0}e^{kt}$$

$$\ln(.3) = \ln\left(e^{kt}\right) = kt$$

$$t = \frac{\ln(.3)}{k} = \frac{\ln(.3)\cdot 5730}{\ln\left(\frac{1}{2}\right)}$$

$$\boxed{t \approx 9950 \text{ years}}$$

10. [10 points] Find the general solution of the following differential equation. You may assume $x \neq 0$.

$$\frac{dy}{dx} \overset{P(x)}{-\frac{2}{x}} y = \overset{Q(x)}{x^2}$$

$$I.F. = e^{\int \frac{-2}{x} dx} = e^{-2\int \frac{1}{x} dx} = e^{-2\ln|x|} = e^{\ln(|x|^{-2})} = |x|^{-2}$$

$$= \frac{1}{x^2}$$

4

5 $$y' \cdot \frac{1}{x^2} - \frac{2}{x} y \cdot \frac{1}{x^2} = x^2 \cdot \frac{1}{x^2}$$

6 $$y' \cdot x^{-2} - 2 y x^{-3} = 1$$

7 $$\left[y \cdot x^{-2}\right]' = 1$$

8 $$\int \left[y \cdot x^{-2}\right]' dx = \int 1 \cdot dx$$

9 $$y \cdot x^{-2} = x + C$$

10 $$\boxed{y(x) = x^3 + Cx^2}$$

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	