4.6 Logarithmic and Exponential Business Applications

Example 1 "Compound Interest"

$$S = P\left(1+\frac{r}{n}\right)^{nt}$$

S = account value
P = principal
r = interest rate
n = # of compoundings / year
t = time (in years)

How long does it take for your initial deposit of P dollars to double if $r = 8\%$ and interest is compounded 4 times annually $(n=4)$?

$$2P = P\left(1+\frac{0.08}{4}\right)^{4t}$$

$$2 = (1+0.02)^{4t}$$

$$2 = (1.02)^{4t} \iff \log_{1.02} 2 = 4t$$

$$\Updownarrow$$

$$\ln 2 = \ln(1.02^{4t})$$

$$\ln 2 = 4t \ln(1.02)$$

$$\frac{\ln 2}{\ln(1.02)} = 4t$$

$$t = \frac{1}{4}\cdot\frac{\ln 2}{\ln 1.02}$$

$$\frac{\ln 2}{\ln 1.02} = 4t$$

$$t = \frac{1}{4}\frac{\ln 2}{\ln 1.02}$$

$$\boxed{t \cong 8.75 \text{ years}}$$

Example 2 "National Debt"

According to data from the Treasury Department, the national debt was about \$930 billion in 1980. By 2007, it was approximately \$9,008 billion or \$9.008 trillion. Assume the debt grows exponentially (with base e), and use the above data to determine the parameters (i.e. the values of A_0 and r) in the model below.

$$\boxed{y(x) = A_0 e^{rx}}$$

Exponential Growth Model

A_0 = initial amount (i.e amt. at $t=0$)

r = growth rate

Solution:

(1) Let x = years since 1980 (1980 is time 0)

$$\Rightarrow y(0) = A_0 e^{r\cdot 0} = A_0 e^0 = A_0 = \$930 \text{ billion}$$

So set $\boxed{A_0 = 930}$ we'll count by billions.

(2) $\begin{array}{r} 2007 \\ -1980 \\ \hline 27 \end{array}$ $\Rightarrow y(27) = 930 e^{r\cdot 27} = 9{,}008$

$$\Rightarrow e^{r\cdot 27} = \frac{9{,}008}{930}$$

$$\Rightarrow r\cdot 27 = \ln\left(\frac{9{,}008}{930}\right)$$

$$\Rightarrow r = \frac{1}{27}\cdot \ln\left(\frac{9{,}008}{930}\right) \approx \boxed{0.0841}$$

→

Therefore our model for national debt growth is:

$$\boxed{y(x) = 930\, e^{0.0841\,x} \quad (\text{in billions})}$$

Use this model to estimate:

(a) The year in which the debt will reach \$25,000 billion.

(b) The predicted national debt in 2050.

(a) $25{,}000 = 930\, e^{0.0841\,x}$ solve for x (years after 1980)

$$e^{0.0841x} = \frac{25{,}00\cancel{0}}{93\cancel{0}}$$

$$\Rightarrow 0.0841x = \ln\left(\frac{2{,}500}{93}\right)$$

$$x = (0.0841)^{-1} \ln\left(\frac{2{,}500}{93}\right)$$

$$x \approx 39.4$$

$$x + 1980 = \begin{array}{r} 1980.0 \\ +\ 39.4 \\ \hline 2019.4 \end{array} \qquad \boxed{\sim 2019}$$

(b) $\begin{array}{r} 2050 \\ -1980 \\ \hline 70 \end{array} \Rightarrow y(70) = 930\, e^{0.0841\cdot(70)}$

$$\cong 335{,}100$$

Debt in 2050 $\approx$ \$335,100 billion

$= \boxed{\$335.1 \text{ trillion}}$

Example 3 "Richter Scale"

The Richter scale is a logarithmic function which assigns a magnitude to the measured intensity:

$$M(I) = \frac{2}{3}\log\left(\frac{I}{I_0}\right)$$

where I_0 is the threshold intensity, or the smallest intensity that can be accurately measured by the seismometer.

(a) If an earthquake measures a magnitude of 4.8, what is the intensity, relative to I_0?

$$4.8 = \frac{2}{3}\log\left(\frac{I}{I_0}\right) \iff \left(\frac{3}{\cancel{2}}\right)\overset{2.4}{\cancel{4.8}} = \log\left(\frac{I}{I_0}\right)$$

$$\iff 7.2 = \log\left(\frac{I}{I_0}\right) \iff 10^{7.2} = \frac{I}{I_0}$$

$$\iff I = 10^{7.2} I_0 \qquad \text{So} \quad I > \underbrace{10 \text{ million}}_{=10^7} \cdot I_0$$

$$I \approx 15{,}800{,}000\, I_0$$

(b) If an earthquake is twice as intense as in part (a), what is its magnitude?

$$M(2\cdot 10^{7.2} I_0) = \frac{2}{3}\log\left(\frac{2\cdot 10^{7.2}\cancel{I_0}}{\cancel{I_0}}\right)$$

$$\approx 5$$

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(c) How much more intensity is measured in a magnitude 7.1 earthquake as compared to a magnitude 4.8 earthquake like in part (a)?

Solution: Let's find the intensity (I) of a magnitude 7.1 earthquake and compare it with the intensity found in part (a).

$$7.1 = \frac{2}{3}\log\left(\frac{I}{I_0}\right) \Rightarrow \frac{3}{2}\cdot(7.1) = \log\left(\frac{I}{I_0}\right)$$

$$10.65 = \log\left(\frac{I}{I_0}\right) \Rightarrow 10^{10.65} = \frac{I}{I_0}$$

$$\Rightarrow \cancel{I = 10.65\, I_0} \quad I = 10^{10.65}\, I_0$$

$$\Rightarrow \frac{I_{7.1}}{I_{4.8}} = \frac{10^{10.65}\cancel{I_0}}{10^{7.2}\cancel{I_0}} = 10^{10.65-7.2} = 10^{3.45} \approx \boxed{2,818}$$