2.3 Gauss – Jordan Elimination

A system of two equations can be translated into a single matrix equation. For example

Ex. 1

$$\begin{cases} x + 8y = -20 \\ 4x + 2y = 10 \end{cases} \iff \begin{bmatrix} 1 & 8 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -20 \\ 10 \end{bmatrix}$$

If we let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -20 \\ 10 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 8 \\ 4 & 2 \end{bmatrix}$

then we further compactify the above matrix equation:

$$\begin{bmatrix} 1 & 8 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -20 \\ 10 \end{bmatrix} \iff A\vec{x} = \vec{v} \quad (*)$$

Notice the similarity with the simple linear equation: $ax = b$ where a & b are constants and x is a variable. We solve this simple equation by multiplying both sides by $\frac{1}{a}$ or equivalently a^{-1}.

$$\frac{1}{a}\left[ax = b\right] \iff x = \frac{b}{a}$$

OR

$$a^{-1}[ax = b]$$
$$a^{-1} \cdot a \cdot x = a^{-1}b$$
$$1 \cdot x = a^{-1}b = \frac{b}{a}$$
$$\Rightarrow \boxed{x = \frac{b}{a}}$$

what if we could do the same thing with the matrix equation $(*)$?

$$A^{-1}\left[A\vec{x} = \vec{v}\right] \iff A^{-1}A\vec{x} = A^{-1}\vec{v}$$
$$\iff I\vec{x} = A^{-1}\vec{v}$$
$$\iff \vec{x} = A^{-1}\vec{v}$$

where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ the "identity matrix"

We can! (Well, under certain circumstances we can.)

So how do we find A^{-1}, the inverse of the matrix A?

The answer is Gaussian elimination or Gauss-Jordan elimination. However before we learn how to find the inverse of a matrix, we'll solve a simpler problem.

Instead of solving the full matrix equation (*), we'll develop a method for solving (*) when we are given a specific $\vec{v}$ vector.

"augmented matrix"

$$\begin{cases} x + 8y = -20 \\ 4x + 2y = 10 \end{cases} \iff \left[\begin{array}{cc|c} 1 & 8 & -20 \\ 4 & 2 & 10 \end{array}\right]$$

Goal: Use the three elementary row operations (below) repeatedly until the left hand side of the augmented matrix equals the 2x2 identity matrix, i.e. $\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right]$ then $x = a$, $y = b$ will be our solution.

Important!

Elementary Row Operations

1. Swap two rows.
2. Multiply any row by a nonzero constant.
3. Add a multiple of one row to another row.

Ex. 1 (continued) First eliminate elements under the diagonal:

5.) $\left[\begin{array}{cc|c} 1 & 8 & -20 \\ 4 & 2 & 10 \end{array}\right] + (-4)R1 \left[\begin{array}{cc|c} 1 & 8 & -20 \\ 0 & -30 & 90 \end{array}\right] R2 * \frac{-1}{30}$

$\left[\begin{array}{cc|c} 1 & 8 & -20 \\ 0 & 1 & -3 \end{array}\right] + (-8)R2 \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -3 \end{array}\right] \Rightarrow \begin{array}{l} x = 4 \\ y = -3 \end{array}$

$-8(-3) + -20 = 4$

check: $1(4) + 8(-3) = 4 + -24 = -20$ ✓
$4(4) + (-3) = 16 + -6 = 10$ ✓

Ex. 2

18.) $\begin{array}{l} 5x + y = 4 \\ 15x + 3y = 21 \end{array} \Rightarrow \left[\begin{array}{cc|c} 5 & 1 & 4 \\ 15 & 3 & 21 \end{array}\right]$

$\left[\begin{array}{cc|c} 5 & 1 & 4 \\ 15 & 3 & 21 \end{array}\right] * \frac{1}{3} \left[\begin{array}{cc|c} 5 & 1 & 4 \\ 5 & 1 & 7 \end{array}\right] + (-R1) \left[\begin{array}{cc|c} 5 & 1 & 4 \\ 0 & 0 & 3 \end{array}\right]$

Second row implies: $0x + 0y = 3$?!

impossible thus <u>No Solution</u>

Ex. 3

33.)

$$7x + 5y - 5z = 16$$
$$3x + 4y + 2z = 17$$
$$3x - 3y + 2z = 10$$

Ans: $x = 3$
$y = 1$
$z = 2$