2.1 Basic Operations with Matrices.

* So far we have studied:

1. linear equations in 1 variable

   e.g. $ax + b = c$

2. linear equations in 2 variables

   e.g. $ax + by = c$

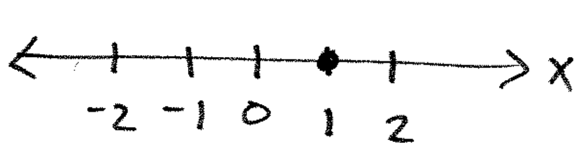
3. linear equations in 3 variables

   e.g. $ax + by + cz = d$

* The graphs or solution sets of these equations were:

1. linear equation in 1 variable:

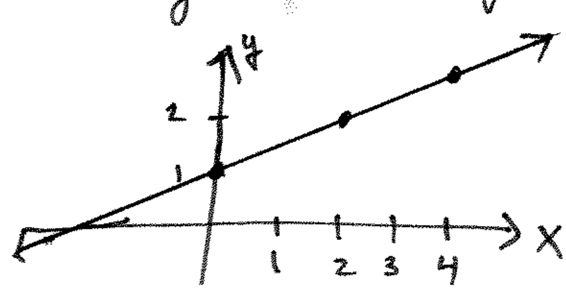
   eg. $2x + 1 = 3 \Rightarrow 2x = 2 \Rightarrow \boxed{x = 1}$



a point!

2. linear equation in 2 variables:

   e.g. $-x + 2y = 2 \Rightarrow 2y = x + 2 \Rightarrow y = \frac{1}{2}x + 1$



$\boxed{y(x) = \frac{1}{2}x + 1}$

a line!

$y : \mathbb{R} \to \mathbb{R}$

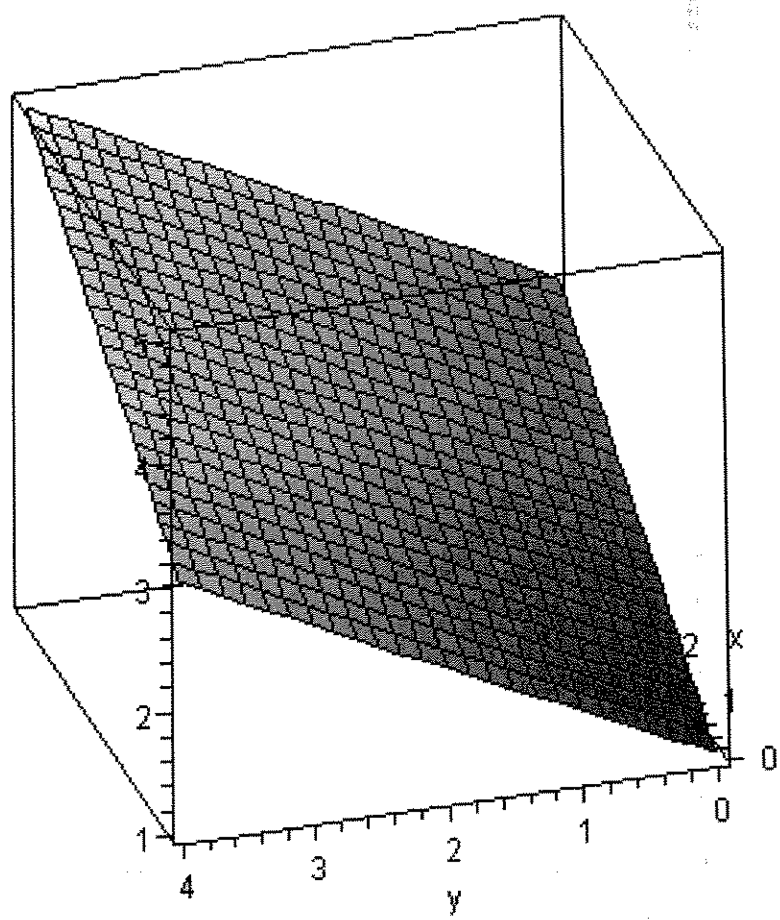
3. linear equation in 3 variables:

e.g. $-\frac{1}{2}x - \frac{1}{2}y + z = 1 \Rightarrow z = \frac{1}{2}x + \frac{1}{2}y + 1$

$$\boxed{z(x,y) = \frac{1}{2}x + \frac{1}{2}y + 1}$$

$$z : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

> plot3d$\left(\frac{1}{2}x + \frac{1}{2}y + 1, x = 0..4, y = 0..4\right)$;



a plane!

We could continue in this manner indefinitely, but the graphs become infeasible. The point is that we can get a linear function in <u>any</u> number of variables e.g. $f: \mathbb{R}^n \longrightarrow \mathbb{R}$

$$f(x_1, x_2, x_3, \ldots, x_n)$$

Next, we studied systems of equations:

$$\begin{aligned} (1)\ & 3x - y = 5 \\ (2)\ & 5x - y = 7 \end{aligned} \implies \begin{array}{r} -3x + y = -5 \\ +\ 5x - y = 7 \\ \hline 2x \quad = 2 \end{array} \implies \boxed{x = 1}$$

plug $x=1$ into (1): $3(1) - y = 5 \implies 3 - y = 5$

$$-y = 2$$

$$\boxed{y = -2}$$

What if we make the algebraic expressions on the left hand sides of equations (1) and (2) into functions?

$$f_1(x,y) = 3x - y$$

$$f_2(x,y) = 5x - y$$

we can create a linear function or <u>map</u>:

$$F(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} 3x - y \\ 5x - y \end{bmatrix}$$

$$F : \mathbb{R}^{2} \longrightarrow \mathbb{R}^{2}$$

input & output are both 2 dimensional!

<u>ex.</u> $F(1,-2) = \begin{bmatrix} f_1(1,-2) \\ f_2(1,-2) \end{bmatrix} = \begin{bmatrix} 3\cdot 1 + 2 \\ 5\cdot 1 + 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

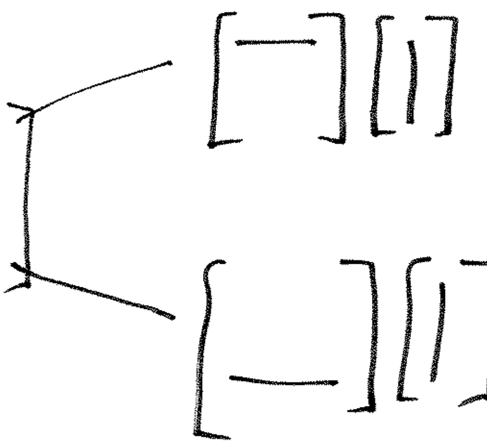
But writing large systems is tedious, let's use a new, more compact notation for functions like F which have <u>vector-valued output</u>:

$$\begin{bmatrix} 3 & -1 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \sim F(x,y)$$

This is F inputs

We get the outputs by matching up rows of F with the input column, then multiply and add, or subtract.

$$\begin{bmatrix} 3 & -1 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x & -y \\ 5x & -y \end{bmatrix}$$



<u>Ex</u> From the previous page,

we know $F(1,-2) = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$, so let's plug in $x=1$, $y=-2$:

$$\begin{bmatrix} 3 & -1 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3\cdot 1 + (-1)(-2) \\ 5\cdot 1 + (-1)(-2) \end{bmatrix} = \begin{bmatrix} 3+2 \\ 5+2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix} \checkmark$$

These outputs match the right-hand side of the equations on the previous page.

We call $F = \begin{bmatrix} 3 & -1 \\ 5 & -1 \end{bmatrix}$ a <u>matrix</u>

The numbers in the grid or matrix are <u>matrix elements</u>. We can refer to them by row number and column number.

<u>Ex</u> $f_{2,1} = 5$

row column

By tradition, we use a capital letter to refer to the whole matrix, and lower-case letters when referring to matrix elements.

<u>Adding Matrices</u>

or <u>order</u>

First, matrices have specific dimensions or <u>size</u>. Here we're referring to the number of rows and columns. So the <u>order</u> of F is 2x2, read "two by two".

<u>Ex</u>. $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 6 \end{bmatrix}$ order = 2×3

$a_{13} = ?$

$B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ order = 4×2

$b_{12} = ?$

Only matrices of the same order or size can be added. Matrices are added element-wise.

Ex

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 0 & 1 \end{bmatrix} \quad 2 \times 3$$

$$B = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \quad 2 \times 3$$

$$A + B = \begin{bmatrix} 1+0 & 3+1 & 5+1 \\ 2+2 & 0+3 & 1+4 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 6 \\ 4 & 3 & 5 \end{bmatrix}$$

Transpose

A^T read "A transpose" is a new matrix formed by changing the rows of A into columns.

Ex. If A is the matrix given above, then:

$$A^T = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 5 & 1 \end{bmatrix} \quad \text{order} = 3 \times 2$$

Notice: order of A was 2×3 so the dimensions traded places.

Ex. 37.) $C = \begin{bmatrix} -2 & 4 \\ -1 & -2 \\ -2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -1 & 2 \\ -2 & 3 & -4 \end{bmatrix}$

Compute: $C + 2B^T$

$$C + 2B^T = \begin{bmatrix} -2 & 4 \\ -1 & -2 \\ -2 & 4 \end{bmatrix} + 2\begin{bmatrix} 3 & -2 \\ -1 & 3 \\ 2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 4 \\ -1 & -2 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} 6 & -4 \\ -2 & 6 \\ 4 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} -2+6 & 4-4 \\ -1-2 & -2+6 \\ -2+4 & 4-8 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -3 & 4 \\ & -4 \end{bmatrix}$$

Ex $A = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$ 3×1 $B = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix}$ 1×3

41.) $A + B$ Not possible! Wrong dimensions!

46.) $B + A^T = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 \end{bmatrix}$