Math 6310, Assignment 3 **Due in class: 19 October, Monday**

1. Let $n \geqslant 3$. Prove that $x^3 = (1\ 2\ 3)$ has no solution in S_n.

2. How many elements of the group S_8 commute with the permutation $(12)(34)(56)$?

3. Let $H(\mathbb{F}_p)$ be the Heisenberg group over $\mathbb{F}_p$ constructed in class as the semidirect product $(\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathbb{Z}/p$, where the automorphism is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_p)$$

Prove that if p is odd, then $H(\mathbb{F}_p)$ has exponent p, and that $H(\mathbb{F}_2)$ is isomorphic to D_4, and so of exponent 4.

4. Let H be a group, K a finite cyclic group and $\varphi_1, \varphi_2 \colon K \longrightarrow \mathrm{Aut}(H)$ homomorphism of groups. Prove that if $\varphi_1(K)$ and $\varphi_2(K)$ are conjugate subgroups of $\mathrm{Aut}(H)$, then there is an isomorphism of groups

$$H \rtimes_{\varphi_1} K \cong H \rtimes_{\varphi_2} K$$

Hint: Suppose $\sigma\varphi_1(K)\sigma^{-1} = \varphi_2(K)$ for some σ in $\mathrm{Aut}(H)$, then there exists an integer a such that

$$\sigma\varphi_1(k)\sigma^{-1} = \varphi_2(k)^a \quad \text{for all } k \in K.$$

The map $H \rtimes_{\varphi_1} K \longrightarrow H \rtimes_{\varphi_2} K$ defined by $(h,k) \longmapsto (\sigma(h), k^a)$ is the desired homomorphism.

5. Let G be a finite group, K a normal subgroup of G, and P a Sylow p-subgroup of K. Prove that $G = KN_P$, where N_P is the normalizer of P in G.

6. Determine, up to isomorphism, all groups of order 99.

7. Determine, up to isomorphism, all groups of order 63.

8. Let G be a finite group, and $\varphi \colon G \longrightarrow G$ a homomorphism.
   (a) Prove that there exists n such that $\mathrm{Image}\,\varphi^m = \mathrm{Image}\,\varphi^n$ and $\ker\varphi^m = \ker\varphi^n$ for all $m \geqslant n$.
   (b) For n as above, prove that G is the semidirect product $(\ker\varphi^n) \rtimes (\mathrm{Image}\,\varphi^n)$.

9. Let G be a finite group, and p a prime dividing $|G|$ such that the map $x \longmapsto x^p$ is a homomorphism.
   (a) Prove that G has a unique Sylow p-subgroup P.
   (b) Prove that there exists $N \lhd G$ such that $N \cap P = \{e\}$ and $G = PN$.
   (c) Show that G has a nontrivial center.

10. Let $|G| = p^k m$ where p is a prime. Let X be the set of p^k-element subsets of G.
    (a) Show that $|X|/m \equiv 1 \bmod p$.
    (b) Let G act on X by left translation, i.e., $g(S) = gS$ for $S \in X$. Prove that the order of each stabilizer subgroup G_S divides p^k. (Hint: G_S acts on S by left translation.)
    (c) Let $Y = \{S \in X : |G_S| = p^k\}$, and show that $|X| \equiv |Y| \bmod pm$.
    (d) Prove that $Y = \{Hx : H \text{ is a subgroup of } G \text{ with } |H| = p^k, \text{ and } x \in G\}$.
    (e) Conclude that the number of subgroups of G of order p^k is 1 mod p.

    This extends the Sylow theorems, since we did not assume m is relatively prime to p.