Solution Manual

4.3

2 $f(z) = \dfrac{1}{z-z^2} = \dfrac{-1}{z} + \dfrac{1}{z-1} = -\dfrac{1}{z} - \dfrac{1}{1-z}$

$= -\dfrac{1}{z} - (1 + z + z^2 + z^3 + \cdots)$

3 $f(z) = z^3\left(1 + \dfrac{1}{z} + \dfrac{1}{2!\,z^2} + \dfrac{1}{3!\,z^3} + \dfrac{1}{4!\,z^4} + \cdots\right)$

$= z^3 + z^2 + \dfrac{z}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!\,z} + \dfrac{1}{5!\,z^2} + \dfrac{1}{6!\,z^3} + \cdots$

4 $f(z) = z^{-3} + z^{-2} + \dfrac{z^{-1}}{2!} + \dfrac{1}{3!} + \dfrac{z}{4!} + \dfrac{z^2}{5!} + \dfrac{z^3}{6!} + \cdots$

6 $f(z) = \dfrac{1}{2}\,\dfrac{1}{z-i} + \dfrac{1}{2}\cdot\dfrac{1}{z+i} = \dfrac{1}{2}\cdot\dfrac{1}{z-i} + \dfrac{1}{2}\cdot\dfrac{1}{2i}\cdot\dfrac{1}{1+\frac{z-i}{2i}}$

$= \dfrac{1}{2}(z-i)^{-1} + \dfrac{1}{4i}\left[1 - \left(\dfrac{z-i}{2i}\right) + \left(\dfrac{z-i}{2i}\right)^2 - \left(\dfrac{z-i}{2i}\right)^3 + \cdots\right]$

7 $f(z) = \dfrac{1}{z} - \dfrac{1}{3!\,z^3} + \dfrac{1}{5!\,z^5} - \dfrac{1}{7!\,z^7} + \cdots$

8 $f(z) = e^z e^{\frac{1}{z}} = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{z^{-m}}{m!}$

$= \displaystyle\sum_{m,n=0}^{\infty} \frac{z^{n-m}}{n!\,m!} = \sum_{i=-\infty}^{\infty} \left(\sum_{j=\max(i,0)}^{\infty} \frac{1}{\left(\frac{i+j}{2}\right)!\left(\frac{j-i}{2}\right)!} \right) z^i$

$i = n-m \qquad (i = -\infty \text{ to } \infty$

$j = n+m \qquad j = 0 \text{ to } \infty)$

\# 10 $a_{k+j} = \frac{1}{2\pi i} \int \frac{f(z)}{z^{k+j+1}} dz$

$|a_{k+j}| \leq \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \frac{|f(z)|}{r^{k+j+1}} ds \quad z = re^{i\theta}$

$\leq \frac{1}{2\pi} \cdot \frac{1}{r^{j+1}} \cdot 2\pi r \underset{\lim_{r\to\infty}}{\rightarrow} 0 \quad \text{if } j \geq 1$

\# 11 $\frac{1}{\sin z} = \frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!}} = \frac{1}{z}\left(\frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots}\right)$

$= \frac{1}{z} \cdot \left(1 + \frac{z^2}{3!} + \left(\frac{1}{36} - \frac{1}{120}\right) z^4 + \cdots\right)$

\# 12 $C_{-1} = 1 \quad C_1 = \frac{1}{6} \quad C_3 = \frac{7}{120 \times 3} = \frac{7}{360}$

4.4

\# 1 $f(z) = \frac{\frac{1}{3}}{2z-1} - \frac{2}{3}\frac{1}{z-2} = \frac{1}{6}\frac{1}{(z-\frac{1}{2})} - \frac{2}{3}\frac{1}{(z-2)}$

$\text{Res}(f, \tfrac{1}{2}) = \frac{1}{6} \quad \text{Res}(f, 2) = -\frac{2}{3}$

\# 2 $f(z) = \frac{A}{z} + \frac{B}{z-3} = \frac{1}{z^2-3z} \quad A(z-3) + Bz = 1$

$A + B = 0$

$\int_\gamma f(z) dz = 0$

\# 3 $\frac{1}{z^2-1} = \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z+1}$

$$\int_{|z|=2} \frac{e^z}{z^2-1}\, dz = 2\pi i\left[\operatorname{Res}(f,1) + \operatorname{Res}(f,-1)\right] \quad (3)$$

$$= 2\pi i\left(\frac{1}{2}e^{+1} - \frac{1}{2}e^{-1}\right)$$

\# 4 $\quad e^z - 1 = z + \frac{z^2}{2!} + \cdots = z\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)$

$$\int_\gamma \frac{1}{e^z-1}\, dz = 2\pi i$$

\# 5

$$\frac{1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

$$1 = A(z-1)(z-2) + B\,z(z-2) + C\,z(z-1)$$

$$A + B + C = 0$$

$$\int \frac{1}{f(z)}\, dz = 0$$

\# 6

$$\frac{1}{z^3-z}(3z^2-1) = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+1}$$

$$A(z-1)(z+1) + B\,z(z+1) + C\,z(z-1) = 3z^2 - 1$$

$$A + B + C = 3$$

$$\int f(z)\, dz = 2\pi i \cdot 3$$

Note. $|0| < 2$

$|1| = |-1| < 2$

\# 8

$$\tan z = \frac{\sin z}{\cos z}$$

$\cos z = 0 \quad |z| < \pi$

$\Rightarrow z = \frac{\pi}{2}, -\frac{\pi}{2}$

$\sin\frac{\pi}{2} = 1 \qquad \sin\left(-\frac{\pi}{2}\right) = -1$

$\cos z = \cos\frac{\pi}{2} + (-\sin\frac{\pi}{2})(z-\frac{\pi}{2})$ + higher order terms.

$= -1\,(z-\frac{\pi}{2}) + \cdots$

$\cos z = \cos(-\frac{\pi}{2}) + (-\sin(-\frac{\pi}{2}))\,(z+\frac{\pi}{2}) + \cdots$

$= (z+\frac{\pi}{2}) + \cdots$

$\int = 2\pi i\,(-1-1) = -4\pi i$

7 8 $|z-5| = 4$ lies in the right half plane

i.e. $\log z$ is analytic

Consider $\sin z = 0$ when $|z-5| < 4$

$z = \pi,\ 2\pi$

$\sin z = \sin\pi + \cos\pi\,(z-\pi)$ + higher order terms

$= -(z-\pi) + \cdots$

$\sin z = \sin 2\pi + \cos(2\pi)\,(z-2\pi)$ + h.o.t.

$= (z-2\pi) + \cdots$

$\int = 2\pi i(-\log\pi + \log 2\pi) = 2\pi i \log 2$

#9 if the interior of γ contains $1, -1$

$\int = (1-1)\,2\pi i = 0$

If the interior contain 1 only, $\int = 1$ $(\mathcal{J})$.

If -1 $\int = -1$

If the int. does not contain 1 or -1

$$\int = 0$$

\#10 $\cot z = \dfrac{\cos z}{\sin z}$ $\sin z = 0$ when z is a multiple of π

$$z = k\pi$$

$$\sin z = \sin(k\pi) + \cos(k\pi)(z - k\pi) + h.o.t.$$

$$\operatorname{Res}(f, k\pi) = \frac{\cos(k\pi)}{\cos(k\pi)} = 1$$

So the integrals counts how many $k\pi$'s are in the interior of γ.

Another view pt: $\displaystyle\int \cot z = \int \frac{(\sin z)'}{\sin z}\, dz$

\#11

$$\frac{z+1}{z-1} = -t \qquad t \geq 0 \qquad t \in \mathbb{R}$$

$$z+1 = -t(z-1)$$

$$z = \frac{t-1}{t+1} \in \mathbb{R} \qquad z(0) = -1 \qquad z(\infty) = 1$$

Solution Manual

4.5 #1

$f(z) = 3z^7 - z^3 + 1$

$g(z) = 3z^7$

$|(f-g)(z)| = |-z^3+1| \leq |z|^3 + 1 = 2 \leq 3 = |g(z)|$

$|f(z)| \geq 3|z|^7 - |z^3| - 1 = 1 > 0$

when $|z| = 1$

$\Rightarrow$ f has 7 zeroes inside as g does

#6 $f(z) = (z-z_0)^k g(z)$

find a δ- neighborhood of z_0 in V

s.t. $|z - z_0| < \delta$ then $|g(z)| \neq 0$

!! Let $m = \min_{\partial D(z_0,\delta)} |f|$

$\forall\ \omega \in D(0,m)$

$h(z) = f(z) - \omega$

$|h(z) - f(z)| = |\omega| \leq m \leq |f(z)|$ where $z \in \partial D(z_0,\delta)$

$f(z) \neq 0$ on $\partial D(z_0,\delta) = \gamma$

$\Rightarrow$ h & f has the same # of zeroes (2).

inside γ

#7 $f'(z) = 6z^2 + 6z - 12$

$$= 6(z^2 + z - 2)$$

$$= 6(z-1)(z+2)$$

when $z \neq 1$ or -2, the local inverse exists.

5.1

#1 $\mathrm{Res}(f, 0) = \frac{2}{-4} = -\frac{1}{2}$

$\mathrm{Res}(f, 1) = \frac{3}{5}$

#2 $\mathrm{Res}(f, 0) = \frac{\cos 0}{2} = \frac{1}{2}$

$\mathrm{Res}(f, 2) = \frac{\cos 2}{2}$

#3 $f(z) = 1 + \frac{1}{z} + \cdots$

$\mathrm{Res}(f, 0) = 1$

#4 $f(z) = \frac{\cos z}{\sin z} = \frac{\cos z}{z - \frac{z^3}{3!} + \cdots}$

$\mathrm{Res}(f, 0) = 1$

(3)

#5 $f(z) = \dfrac{\cos z}{\sin z}$

$$= \frac{\cos z}{-\sin(z-\pi)} = \frac{\cos z}{-(z-\pi) + \frac{(z-\pi)^3}{3!} + \cdots}$$

$\operatorname{Res}(f, \pi) = \dfrac{\cos \pi}{-1} = 1$

#6 I did it in class !!

Cot is even $\operatorname{Res}(f, 0) = 0$

#7 $f(z) = \dfrac{1 + z + \frac{z^2}{2} + \cdots}{-\frac{z^2}{2} + \frac{z^4}{4!} + \cdots} = \dfrac{-2}{z^2} \dfrac{1 + z + \frac{z^2}{2} + \cdots}{(1 - \frac{z^2}{24} + \cdots)}$

$\operatorname{Res}(f, 0) = -2$

#8 $f(z) = \dfrac{1}{z^3 - \frac{z^5}{3!} + \cdots} = \dfrac{1}{z^3(1 - \frac{z^2}{6} + \cdots)}$

$$= \frac{1}{z^3}\left(1 + \frac{z^2}{6} + \cdots\right)$$

$\operatorname{Res}(f, 0) = \dfrac{1}{6}$

#9 $f(z) = \dfrac{1}{\frac{x^2}{2} - \frac{x^3}{3} + \cdot} = \dfrac{1}{\frac{x^2}{2}(1 - \frac{2x}{3} + \cdot)} = \dfrac{2}{x^2}\left(1 + \dfrac{2x}{3} + \cdot\right)$

$\operatorname{Res}(f, 0) = \dfrac{4}{3}$

Solution Manual

5.2

#1 $\cos\theta = \dfrac{z+z^{-1}}{2}$ $\quad z = e^{i\theta} \quad dz = i e^{i\theta} d\theta$

$$\int_\gamma \frac{1}{5 - 4 \cdot \frac{z+z^{-1}}{2}} \cdot \frac{1}{iz}\, dz = i \int \frac{1}{2z^2 - 5z + 2}\, dz$$

$$= i \cdot 2\pi i \cdot \operatorname{Res}(f, \tfrac{1}{2}) = -2\pi \cdot \frac{1}{2(\frac{1}{2} - 2)} = \frac{2}{3}\pi$$

#2 $\sin\theta = (z - z^{-1})/2i$

$$\int_\gamma \frac{1}{10 + 6(z - z^{-1})/2i} \cdot \frac{1}{iz}\, dz = \int_\gamma \frac{1}{3z^2 + 10iz - 3}\, dz$$

$$= 2\pi i \operatorname{Res}(f, \tfrac{i}{3}) = 2\pi i \cdot \frac{1}{3(\frac{i}{3} - 3i)} = -\frac{1}{4}\pi$$

#3
$$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}\, dx = 2\pi i \operatorname{Res}\left(\frac{1}{z^2+2z+2}, -1+i\right)$$

$$= 2\pi i \frac{1}{(-1+i) - (-1-i)} = \pi$$

#4
$$2\int_0^{\infty} \frac{x^2}{(1+x^2)^2}\, dx = \int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2}\, dx = 2\pi i \operatorname{Res}(f, i)$$

$$\int_0^{\infty} \frac{x^2}{(1+x^2)^2}\, dx = \pi i \cdot -\frac{i}{4} = \frac{\pi}{4}$$

#5 $\int_{-\infty}^{\infty}$ odd fcn $= 0$

6 $2\int_0^\infty \frac{1}{1+x^6}\,dx = 2\pi i \sum_{\substack{z=i\\ e^{i\frac{\pi}{6}}}} Res(f, z)$ (2).

$\int_0^\infty \frac{1}{1+x^6}\,dx = \pi i\, [\frac{1}{2i\cdot(1+1+1)} + \frac{1}{6(e^{i\frac{\pi}{6}})^5} + \frac{1}{6(e^{i\frac{5\pi}{6}}$

$(1+z^6) = (z^2+1)(z^4-z^2+1) = 1+[(z-z_0)+z_0]^6$

$= (z_0^6+1) + 6z_0^5(z-z_0) + h.o.t$

$= \pi i\, [-\frac{i}{6} + -\frac{e^{i\frac{\pi}{6}}}{6} + -\frac{e^{i\frac{5\pi}{6}}}{6}] = \pi i \cdot \frac{1}{6}(-2$

$= \frac{\pi}{3}$

7 $\int_{t=0}^\infty \frac{1}{1+z^3}\,dz \qquad z = e^{i\frac{2\pi}{3}}t$

$dz = e^{i\frac{2\pi}{3}}\,dt$

$= e^{i\frac{2\pi}{3}} \int_{t=0}^\infty \frac{1}{1+t^3}\,dt$

i.e. $(1 - e^{i\frac{2\pi}{3}}) \int_{t=0}^\infty \frac{1}{1+t^3}\,dt = 2\pi i\, Re(\frac{1}{1+z^3}, z_0)$

$z_0 = \frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\frac{\pi}{3}}$

$\int_{t=0}^\infty \frac{1}{1+t^3}\,dt = \frac{1}{3z_0^2}\ \frac{1}{1-z_0^2} \cdot 2\pi i = \frac{2}{3}\pi i \cdot \frac{1}{2i\frac{\sqrt{3}}{2}}$

$= \frac{2\pi}{3\sqrt{3}}$

(3).

#8 $\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx$

$\left| \frac{e^{i(x+iy)}}{1+z^2} \right| \leq \frac{e^{-y}}{R^2-1} < \frac{1}{R^2-1}$ since $y > 0$

$= 2\pi i \operatorname{Res}\left(\frac{e^{iz}}{1+z^2}, i \right)$

$= 2\pi i \cdot \frac{e^{-1}}{2i} = \frac{\pi}{e}$

#14 $\int_0^{\infty} \frac{1}{t + t(\log t)^2}\,dt = \int_{x=-\infty}^{\infty} \frac{1}{1+x^2}\,dx = 2\pi i \cdot \frac{1}{2i} = \pi$

$x = \log t$

5.3

#2 $\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x}{1+x^2} e^{-itx}\,dx$

$= \sqrt{2\pi}\, i \operatorname{Res}\left(\frac{z}{1+z^2} e^{-itz}, i \right)$ if $t < 0$

$= \sqrt{2\pi}\, i \cdot \frac{i}{2i} \cdot e^{t} = \sqrt{\frac{\pi}{2}}\, e^{t}\, i$

$\hat{f}(t) = -\sqrt{2\pi}\, i \operatorname{Res}\left(\frac{z}{1+z^2} e^{-itz}, -i \right)$ if $t > 0$

$= -\sqrt{2\pi}\, i \cdot \frac{(-i)}{\quad} e^{-t} = -\sqrt{\frac{\pi}{2}}\, e^{-t}\, i$

3 $\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1+x^4} e^{-itx}\, dx$ (3 ... = (4))

$$\hat{f}(t) = \sqrt{2\pi}\, i \sum_{z_0} \mathrm{Res}\left(\frac{1}{1+z^4} e^{-itz}, z_0 \right) \quad \text{when } t < 0$$

$$[= e^{i\frac{\pi}{4}}, e^{+i\frac{\pi}{4}3}]$$

$$= \sqrt{2\pi}\, i \sum \frac{1}{4 z_0^3} e^{-itz_0}$$

$$\hat{f}(t) = -\sqrt{2\pi}\, i \sum \frac{1}{4 z_0^3} e^{-iz_0 t} \quad \text{when } t > 0$$

$$z_0 = e^{-i\frac{\pi}{4}}, e^{-i\frac{\pi}{4}3}$$

4 $\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{2}}\, e^{-|x|} e^{-itx}\, dx$

$$= \frac{1}{2} \int_{-\infty}^{0} e^{x-itx}\, dx + \frac{1}{2} \int_{0}^{\infty} e^{-x-itx}\, dx$$

$$= \frac{1}{2} \frac{1}{1-it} e^{(1-it)x} \Big|_{-\infty}^{0} + \frac{1}{2} \frac{1}{-(1+it)} e^{-(1+it)x} \Big|$$

$$= \frac{1}{2} \cdot \frac{1}{1-it} + \frac{1}{2} \cdot \frac{1}{1+it} = \frac{1}{1+t^2}$$

#5

$$\hat{f}(t) = \sqrt{2\pi}\, i \ \mathrm{Res}\left(\frac{1}{z^2+4z+5} \cdot e^{-itz}, -2+i\right) \quad \text{when } t < 0$$

$$= \sqrt{2\pi}\, i \ \frac{1}{2i} e^{(1+2i)t}$$

$$= \sqrt{\frac{\pi}{2}}\, e^{(1+2i)t}$$

$$\hat{f}(t) = -\sqrt{2\pi}\, i \cdot \frac{1}{-2i} e^{(-1+2i)t} \quad \text{when } t > 0$$

$$= \sqrt{\frac{\pi}{2}}\, e^{(-1+2i)t}$$

#7

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2}\, dx = \int_{-\infty}^{\infty} \frac{x\left(\frac{e^{ix}-e^{-ix}}{2i}\right)}{1+x^2}\, dx$$

$$= \frac{1}{2i}\left[\int_{-\infty}^{\infty} \frac{x e^{ix}}{1+x^2}\, dx - \int_{-\infty}^{\infty} \frac{x e^{-ix}}{1+x^2}\, dx\right]$$

$$= \frac{1}{2i} \mathrm{Res}\left(\frac{z e^{iz}}{1+z^2}, i\right) \cdot 2\pi i + \frac{1}{2i}\, 2\pi i\, \mathrm{Res}\left(\frac{z e^{-iz}}{1+z^2}, -i\right)$$

if z is on the upper half plane

$$\frac{|z e^{iz}|}{|1+z^2|} \leq \frac{R e^{-y}}{R^2-1} \qquad z = x+iy$$

Similarly for the lower half

$$\#7 \quad = \pi \cdot \frac{i}{2i} e^{-1} + \pi \cdot \frac{-i}{-2i} \cdot e^{-1} = \frac{\pi}{e} \qquad (6)$$

$$\#10 \quad \int_0^\infty \frac{x^{\frac{2}{3}}}{1+x}\,dx$$ uses Mellin Transform, Thm 5.3.7

or # 5.3.8

5.4

#2

$$f(z) = \frac{1}{1+z^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2}\left(\sum_{-\infty}^{\infty} \frac{1}{1+n^2} - 1\right) = \frac{e^{-\pi}}{e^{\pi} - e^{-\pi}} = \frac{1}{e^{2\pi} - 1}$$

$$\sum_{-\infty}^{\infty} \frac{1}{1+n^2} = \sum_{z_0 = \pm i} \operatorname{Res}\left(\frac{1}{1+z^2} \frac{\pi \cos \pi z}{\sin \pi z}, z_0\right)$$

$$= \frac{1}{2i} \frac{e^{-\pi} + e^{\pi}}{e^{-\pi} - e^{\pi}} \cdot i - \frac{1}{2i} \frac{e^{-\pi} + e^{\pi}}{e^{\pi} - e^{-\pi}} \cdot i$$

$$\sum_{-\infty}^{\infty} \frac{1}{1+n^2} = \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}}$$

#4

$$f(z) = \frac{1}{z^4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = -\operatorname{Res}\left(\frac{1}{z^4} \frac{\pi \cos \pi z}{\sin \pi z}, 0\right) = \frac{\pi\left(1 - \frac{\pi^2 z^2}{2} + \frac{\pi^4 z^4}{24} + \cdots\right)}{\pi z^5\left(1 - \frac{\pi^2 z^2}{6} + \frac{\pi^4 z^4}{120} + \cdots\right)}$$

$$= -\operatorname{Res}\left(\frac{1}{z^5}\left(1 - \frac{\pi^2}{3} z^2 - \frac{\pi^4}{45} z^4\right)\right) = \frac{\pi^4}{45}$$

Homework Solutions

6.1

#1 $\begin{bmatrix} x \\ y \end{bmatrix} \to (a+ib)(x+iy) = \begin{bmatrix} ax-by \\ bx+ay \end{bmatrix}$

$$= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$a = r\cos\theta \qquad b = r\sin\theta$

#2 $g(z)+1 = \dfrac{1+z}{1-z} + 1 = \dfrac{2}{1-z}$

#3 how about $\dfrac{1}{z}$?

#4 $\sqrt[4]{i \dfrac{1+z}{1-z}}$

#5 $\dfrac{1+z}{1-z} \in$ Right half plane

need to prove it is in the upper half plane

as well

$$\frac{(1+z)(1-\bar{z})}{|1-z|^2} = \frac{1 - z\bar{z} + (z-\bar{z})}{|1-z|^2} = \frac{1-|z|^2 + 2i\,\mathrm{Im}\, z}{|1-z|^2}$$

$$\frac{2\,\mathrm{Im}\, z}{|1-z|^2} > 0$$

#6 open first quadrant $\xleftarrow{\frac{1+z}{1-z}}$ upper half dis

$\sqrt{i\dfrac{1+z}{1-z}} \quad \uparrow \downarrow \quad \dfrac{i\omega^2+1}{i\omega^2-1} \qquad \swarrow \qquad \dfrac{i\left(\frac{1+z}{1-z}\right)^2 + 1}{i\left(\frac{1+z}{1-z}\right)^2 - 1}$

#6.1.6 unit disc

#7 How about $z^{\frac{a}{\pi}}$?

$\exp\left(\frac{a}{\pi} \log_I z \right)$

#8 Similar to 7.

$a = 2\pi$ $\quad z^2$ maps upper half plane

to $\mathbb{C} \setminus [0, \infty)$

Answer: $-z^2 + 1$

#2 $\dfrac{(z-1)\,\cancel{} - 1}{(z-1)+1} - 1$

#3 $|z| = 1$ under $h(z)$ is a line passing ∞

$h(i) = 2i(i) / (i-1) = -i-1$

$h(-1) = -2i/(-2) = i$

$z - i = t(-2i-1)$

$z = (-t, -2t+1) = (s, 2s+1)$

$h(0) = 0 \Rightarrow$ unit disc goes to right half (lower)

#4 ∂B passes thru $z=1$

the circle goes to a line

$h(i) = \dfrac{1+i}{1-i} = i$

$h(2+i) = \dfrac{1}{2}(-3+i)$

the line is $z - i = t\frac{1}{2}(-3-i)$

$z = -\dfrac{3}{2}t + (-\dfrac{1}{2}t+1)$

$h(1+i) = \dfrac{1+i+1}{-i} = -1+2i \Rightarrow$ left (upper) half

#5 it touches $z = 1$

$\partial B \Rightarrow$ line

$h(0) = 1$

$h(\frac{1}{2} + i\frac{1}{2}) = 1 + 2i$

$z - 1 = t\ (2i)$

$z = 1 + 2t\ i$

$h(\frac{1}{2}) = 3 \Rightarrow$ right half plane

#6 the circle is centered at $1 \pm ai$ w/

radius $= a > 0$, passing thru $z = 1$

$h(1 + 2ai) = \dfrac{2 + 2ai}{-2ai} = \frac{1}{a} i - 1 \qquad h(0) = 1$

$h(1 + a + ai) = \dfrac{2 + a + ai}{-a \pm ai} = +\frac{1}{a}(-1 + i) - 1$

$z - \left(\frac{1}{a} i - 1\right) = t\ \left(-\frac{1}{a}\right)$

$z = -\frac{1}{a} t - 1 + i\frac{1}{a} \Rightarrow$ lower half plane

Now my $\frac{1}{a}$ is the "a" in the textbook.

do it for the $1 - ai$ form

7

$a_1 \to 0 \leftarrow a_2$

$b_1 \to \infty \leftarrow b_2$

$c_1 \to 1 \leftarrow c_2$

$a_1, b_1, c_1 \ni$ x-axis

$z_0 = \frac{1}{3}(a_1+b_1+c_1) \in$ ~~Center~~ inside

$$f(z) = \frac{c_1 - \overset{b_1}{\cancel{a_1}}}{c_1 - a_1}\,\frac{z-a_1}{z-b_1} \qquad f(z_0) = \frac{c_1-b_1}{c_1-a_1}\,\frac{(c_1-a_1)+(b_1-a_1)}{(a_1-b_1)+(c_1-b_1)}$$

$$= \frac{1 + \frac{b_1-a_1}{c_1-a_1}}{1 + \frac{a_1-b_1}{c_1-b_1}}$$

want to check $\operatorname{Im} f(z_0) > 0$ or not

9

$$\frac{\frac{z-\omega}{1-\bar{\omega}z} + \omega}{1 + \bar{\omega}\frac{z-\omega}{1-\bar{\omega}z}} = \frac{(z-\omega)+\omega(1-\bar{\omega}z)}{1-\bar{\omega}z+\bar{\omega}(z-\omega)}$$

$$= \frac{z-\omega+\omega-\omega\bar{\omega}z}{1-\bar{\omega}z+\bar{\omega}z-\omega\bar{\omega}} = \frac{(1-\omega\bar{\omega})z}{1-\omega\bar{\omega}} = z$$

#10

$$h_\omega(z) = \frac{z - \frac{1}{2}}{1 - \frac{1}{2}z}$$

$$u h_\omega(z) = u \frac{z - \frac{1}{2}}{1 - \frac{1}{2}z}$$

$$[u h_\omega(z)]' = u \frac{(1 - \frac{1}{2}z) + \frac{1}{2}(z - \frac{1}{2})}{(1 - \frac{1}{2}z)^2} \Big|_{z=0}$$

$$= u \frac{3}{4} = \frac{3}{4} i$$

$$u = i$$

#11 by observation

$$\frac{1}{2} \to 0 \to \frac{i}{3} = -\left(\frac{-i}{3}\right)$$

$$h_\omega(z) = \frac{z - \frac{1}{2}}{1 - \frac{1}{2}z}$$

$$u h_\omega(z) = u \frac{z - \frac{1}{2}}{1 - \frac{1}{2}z} = g(z) \qquad g\left(\tfrac{1}{2}\right) = 0$$

$$v \frac{\left(\frac{u(z - \frac{1}{2})}{1 - \frac{1}{2}z}\right) - \left(\frac{-i}{3}\right)}{1 - \frac{i}{3}\left(\frac{u(z - \frac{1}{2})}{1 - \frac{1}{2}z}\right)} = h(g(z))$$

$$\frac{d}{dz} h(g(z)) \Big|_{z=\frac{1}{2}} = h'(0)\, g'\left(\tfrac{1}{2}\right) = \left[1 - \left|\tfrac{-i}{3}\right|^2\right] \frac{1}{1 - \left(\frac{1}{2}\right)^2} > 0$$