Solution Manual

2.2

\# 7 $\frac{d}{dz} e^{z^3} = e^{z^3} \cdot 3z^2$

\# 8 For any I of length 2π

$\log_I z$ is analytic on $\mathbb{C} \setminus \{0\} \cup$ cut-line

$\log_I z$ & $\log_{(-\pi,\pi]} z$ differs by a const.

So # 2.2.21 applies.

The derivative $= \dfrac{1 - \log_I z}{z^2}$ w/ the same

domain, cut-lines excluded

\# 11 $f(x,y) = u + iv$ $v = 0$

$\left.\begin{array}{l} u_x(x,y) = 0 \\ u_y(x,y) = 0 \end{array}\right)$ no dependence on x, y

$f(x,y) =$ a constant

\# 13 $\log_I z = \ln r + i(\theta + k)$

$u_r = \frac{1}{r}$ $u_\theta = 0$ $v_r = 0$ $v_\theta = 1$

$\quad \shortparallel \qquad\qquad \shortparallel$

$\frac{1}{r} v_\theta$ $-r v_r$ (according to # 12)

11 $|R(z)| \leq \sum_{k=n+1}^{\infty} \frac{1}{k!}$

$$= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots$$

$$= \frac{1}{(n+1)!} \left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots \right)$$

$$\leq \frac{1}{(n+1)!} \left(1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \cdots \right)$$

$$= \frac{1}{(n+1)!} (e - 1)$$

2.5

3 $f(z) = |z|$ is cont. on K, cpt

$$\exists z_0 \quad f(z_0) = \min_{z \in K} f(z)$$

$$f(z_0) \leq f(z) \Rightarrow |z_0| \leq |z|$$

5 $\int_\gamma z^n \, dz = 0$

6 follows from # 5

7 $\int_{\theta = -\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{e^{i\theta}} i e^{i\theta} \, d\theta = i\pi = \log z \Big|_{-i}^{i}$

8 The loop consists of no holes.

#9 $\int_{\theta=+\pi}^{0} e^{i\theta/2}\, i e^{i\theta}\, d\theta \quad \gamma = e^{i\theta} \quad \theta = +\pi$ to 0 (3)

$= i e^{\frac{3}{2} i\theta} / (\frac{3}{2} i) \Big|_{\pi}^{0} = \dfrac{e^{\frac{3}{2} i\theta}}{\frac{3}{2}} \Big|_{\pi}^{0}$

$= [1 - (-i)] / (\frac{3}{2}) = \frac{2}{3} z^{\frac{3}{2}} \Big|_{-1}^{1}$

#11 $-\int_0^1 z^n\, dz = -\frac{1}{n+1}$

2.6

#1 Thm. 2.6.1 or use the modulus for Cor. 2.6.8

#2 $\int \frac{1}{z^2-4} = \frac{1}{4}\int \frac{1}{z-2} - \frac{1}{z+2}\, dz = 0$

#3 $\int_\gamma \frac{1}{1-e^z} dz = 0$

$1 - e^z = 0$ when $z = 2\pi i k \quad k \in \mathbb{Z}$

#4 $\int_\gamma \frac{1}{z}\, dz = 2\pi i$ if γ, the interior of circle contains 0

$\{ 0$ if not

#5 $\int_{\theta=0}^{2\pi} e^{-2i\theta}\, i e^{i\theta}\, d\theta = 0$

#6 nothing but $\frac{d}{dz} \log z = \frac{1}{z}$

10. $ind_\gamma(z_0) = n$ if $\gamma = z_0 + e^{int}$

11. $Ind_\gamma(1+i) = 1$

$Ind_{-\gamma}(1+i) = -1$

3.1

#1 $\lim_{n\to\infty} \frac{1}{nz} = 0$ if $z \neq 0$

$\left|\frac{1}{nz}\right| \leqslant \frac{1}{nr}$ if $|z| \geqslant r$

pick $z_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n z_n} = 1 \neq 0$

2 $\lim \sin \frac{x}{n} = 0$

$\left|\sin\frac{x}{n}\right| \leqslant \left|\frac{x}{n}\right| \leqslant \frac{k}{n}$

Pick $x_n = n \cdot \frac{\pi}{2}$

$\lim \sin \frac{x_n}{n} = 1 \neq 0$

#3 $\lim \tan^{-1} nx = \begin{cases} \frac{\pi}{2} & x > 0 \\ 0 & x = 0 \\ -\frac{\pi}{2} & x < 0 \end{cases}$ not a cont. fcn

#5 $\left|\sum_{k=1}^{\infty} \frac{k+z}{k^3+1}\right| \leqslant \sum_{k=1}^{\infty} \frac{k+1}{k^3} \leqslant \sum_{1}^{\infty} \frac{2k}{k^3} < \infty$

#6 $\left|\sum_{k=1}^{\infty} \frac{1}{k^2 - z}\right| \leq \sum_{k=1}^{\infty} \frac{1}{|k^2 - z|} \leq \sum_{1}^{\infty} \frac{1}{k^2 - r}$

$\leq \sum_{1}^{n} \frac{1}{k^2 - r} + \sum_{n+1}^{\infty} \frac{1}{k^2 - \frac{1}{2}k^2} < \infty$

#7 $\left|\sum_{1}^{\infty} k^{-z}\right| \leq \sum_{1}^{\infty} \frac{1}{|k^z|} = \sum_{1}^{\infty} \frac{1}{k^x} < \sum_{1}^{\infty} \frac{1}{k^s}$

if $z = x + iy$

#8 $\left(\sum_{k=1}^{\infty} k^{-z}\right)' = \sum_{k=1}^{\infty} e^{-\ln k\, z}$

$= \sum_{k=1}^{\infty} -\ln k \; k^{-z} = -\sum_{k=1}^{\infty} \frac{\ln k}{k^z}$

$\left|\sum_{k=1}^{\infty} \frac{\ln k}{k^z}\right| \leq \sum \frac{\ln k}{k^x} < \sum \frac{\ln k}{k^s} = \sum \frac{\ln k}{k^{\frac{1+s}{2}} k^{\frac{s-1}{2}}} < \infty$

$\lim \frac{\ln k}{k^{\frac{s-1}{2}}} = 0$

#14 No it can not div. at $z = 0$

$|z - 1| = |0 - 1| = 1 <$

$|3 - 1| = 2$

\# 15 $\frac{1}{1+\omega} = 1 - \omega + \omega^2 - \omega^3 + \omega^4 - \omega^5 + \cdots$

$$\int_0^z \frac{1}{1+\omega}\, d\omega = \omega - \frac{1}{2}\omega^2 + \frac{1}{3}\omega^3 - \frac{1}{4}\omega^4 + \frac{1}{5}\omega^5 - \frac{1}{6}\omega^6 + \cdots \Big|_0^z$$

$$= \log(1+z)$$

$R = 1$

\# 16 $E(z) = \int_0^z 1 - \omega^2 + \frac{\omega^4}{2} - \frac{\omega^6}{6} + \frac{\omega^8}{24} - \frac{\omega^{10}}{120} + \cdots$

$$= \omega - \frac{\omega^3}{3} + \frac{\omega^5}{10} - \frac{\omega^7}{42} + \frac{\omega^9}{216} - \frac{\omega^{11}}{1320} \Big|_0^z$$

It conv. everywhere.

Solution Manual

3.2

#1 $$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \cdots$$

$$\frac{1}{(1-z)^2} = \frac{d}{dz}\frac{1}{(1-z)} = 1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots$$

#2 $$(1+z)^{\frac{1}{2}} = 1 + \frac{\frac{1}{2}}{1}\cdot z + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2} z^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!} z^3$$

$$+ \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!} z^4 + \cdots$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2(n+1)-3}{2(n+1)}\right| = 1$$

$$R = 1$$

#4 for each $z \in \partial D_R(z_0)$, $\exists\, r(z)$ $D_{r(z)}(z) \subset U$

We can choose $r(z)$ is cont. in z

$$\min_{z\in\partial D_R(z_0)} |r(z)| = r \qquad \text{therefore} \quad \bar{D}_R(z_0) \subset D_r(z_0) \subset U$$

#7 $$\frac{1}{i-(z-i)} = \frac{1}{z} = \frac{-i}{1-(1+iz)} = -i + (-i)^2(z-i) + \frac{(-i)^3(z-i)^2}{2} + \frac{(-i)^4(z-i)^3}{6} + \frac{(-i)^5(z-i)^4}{24} + \cdots$$

$$\frac{d}{dz}\log z = \frac{1}{z}$$

$$\frac{d}{dz}\log z = -i - (z-i) + \frac{i}{2}(z-i)^2 + \frac{(z-i)^3}{6} - i\frac{(z-i)^4}{24} + \cdots \quad (2)$$

$$\log z = i\frac{\pi}{2} - i(z-i) - \frac{1}{2}(z-i)^2 + \frac{i}{6}(z-i)^3 + \frac{(z-i)^4}{24} - \frac{i}{120}(z-i)^5 + \cdots$$

#8 $\sin z = z - \frac{z^3}{6} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

Cont. extension $f(0) = 1$

#10 $|f'(z_0)| \leq \frac{M}{R}$ according to the theorem, $n = 1$

#11 $|f^{(3)}(z_0)| \leq \frac{6M}{R^3}$

$p^{(3)}(z) = 6a_3$

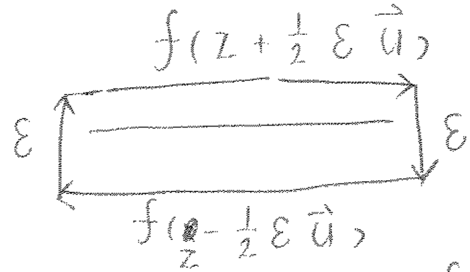
$|6a_3| \leq \frac{6 \cdot 1}{1} \Rightarrow |a_3| \leq 1$

#14 f is cont. & analytic on $U \setminus E$

$|f|$ achieves its max on E (cpt)

\# 14 So we can pass to the ε-neighbood of the line (3.

Seg.



the length of the seg. $= L$

f is cont. So $\sup_{\square} |f|$ exists $= M$

$$\int_{\uparrow} f\,dz + \int_{\rightarrow} f(z+\tfrac{1}{2}\varepsilon\vec{u})\,dz + \int_{\downarrow} f\,dz$$

$$+ \int_{\leftarrow} f(z-\tfrac{1}{2}\varepsilon\vec{u})\,dz$$

$$= \int_{\uparrow} f\,dz + \int_{\downarrow} f\,dz + \int_{\rightarrow} f(z+\tfrac{1}{2}\varepsilon\vec{u}) - f(z-\tfrac{1}{2}\varepsilon\vec{u})\,dz$$

$$\left|\int_{\square} f\,dz\right| \leq 2\varepsilon M + \sup\left|f(z+\tfrac{1}{2}\varepsilon\vec{u}) - f(z-\tfrac{1}{2}\varepsilon\vec{u})\right| L$$

$$\leq 2\varepsilon M + \varepsilon_1 L \quad \text{(by uniform cont.)}$$

$$\to 0$$

\# 16

$$\int_{\partial\Delta} f(z)\,dz = \int_{\partial\Delta}\int_a^b g(z,t)\,dt\,dz$$

$$= \int_a^b \int_{\partial\Delta} g(z,t)\,dz\,dt = 0$$

$|g(z,t)|$ on $\bar{\Delta} \times [a,b]$ is bdd

So you can interchange the integral

3.3 (4)

#1 need to prove $\forall \varepsilon > 0 \quad \exists R \quad$ s.t. $|\frac{1}{z}| < \varepsilon$ when $|z| > R$

just choose $R > \frac{1}{\varepsilon}$

#2 $\lim_{z \to \infty} f(z) = \infty$ then $\forall \varepsilon > 0 \quad K = \frac{1}{\varepsilon} \quad \exists M = R$

s.t. $|z| > M = R \qquad |f(z)| > \frac{1}{\varepsilon} \Rightarrow |\frac{1}{f(z)}| < \varepsilon$

#3 if $f(z) \neq 0 \quad \forall z$

$$\lim_{z \to \infty} \frac{1}{f(z)} = 0$$

$\frac{1}{f(z)}$ analytic & bdd $\Rightarrow$ $f(z)$ is const

$\rightarrow\leftarrow$ since $\lim_{z \to \infty} f(z) = \infty$

#4 $f(z) \neq 0 \quad \forall z$

$|\frac{1}{f(z)}| \leq 1 \qquad \frac{1}{f(z)} = C \qquad f(z) = \frac{1}{C}$

#5 $\frac{f(z) + \overline{f(z)}}{2} < M$

$f = u + iv$ construct $g = (u - M - 1) + iv$

g is entire $|g| \geq 1$ by #4, g is const.

\# 6 by contradiction. $\exists z_0, r$ s.t.

$f(\mathbb{C}) \cap D_r(z_0) = \phi$ i.e. $|f(z) - z_0| > r$

Construct $g(z) = (f(z) - z_0)/r$

$|g(z)| \geq 1$

\# 7 $f(0) = 0$

Construct $g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$

$g(z)$ entire $|g(z)| \leq K \Rightarrow g(z) = C$

$f(z) = Cz$

\# 8 $\frac{f(z)}{e^z}$ is entire since $e^z \neq 0$

$\left|\frac{f(z)}{e^z}\right| \leq K \Rightarrow \frac{f(z)}{e^z} = C$ $f(z) = Ce^z$

\# 9 $|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}$

$|f^{(n+1)}(z_0)| \leq \frac{(1+n)!}{R^{n+1}}(A + BR^n) = \frac{(n+1)!A}{R^{n+1}} + \frac{B}{R} \to C$

#10

$f(\omega)=0 \qquad a_n\omega^n + \cdots + a_0 = 0$

$a_n\bar{\omega}^n + \cdots + a_0 = 0$

$\Rightarrow f(\bar{\omega}) = 0$

#16

$|f^{(n)}(0)| \leqslant \dfrac{n!\,M}{R^n}$

$|f^{(n)}(0)| \leqslant \dfrac{n!\,(A+B\log|z|)}{|z|^n} < \varepsilon$

$f^{(n)}(0) = 0 \quad \text{if} \quad n \geqslant 1$

3.4
#3

$f(0) = \lim_{z\to 0} f(z) = 0$ if f is analytic

$\{\frac{1}{n}\}_{n\in\mathbb{N}} \cup \{0\}$ is not discrete.

Need to construct a f, not continuous at $z=$

$\dfrac{1}{e^{\frac{2\pi i}{z}}-1}$ will do.

#4 $\sin z - z = -\frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdot$ (7)

order $= 3$

#5 $\cos z - 1 + \frac{z^2}{2} = +\frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} + \cdots$

Order $= 4$

#7 if not, $\exists$ infinitely zeroes, z_k

$\{z_k\} \subseteq K$ cpt

$\exists \{z_{k_i}\} \subseteq K$ s.t. $\lim z_{k_i} = z_0 \in K$

$\{z_{k_i}\} \cup \{z_0\}$ non-discrete

#8 $f = (z-z_0)^k \, h(z)$ $\quad h(z) \neq 0$ in V

$= (z-z_0)^k \, e^{\hat{h}(z)}$

$= (z-z_0)^k \left[e^{\frac{1}{k}\hat{h}(z)} \right]^k$

$g(z) = (z-z_0)\, e^{\frac{1}{k}\hat{h}(z)} = (z-z_0)\, h(z)^{\frac{1}{k}}$

$g'(z) = h(z)^{\frac{1}{k}} + (z-z_0) \frac{d}{dz}[h(z)^{\frac{1}{k}}]\Big|_{z=z_0}$

$g'(z_0) = h(z_0)^{\frac{1}{k}} \neq 0$

(8)

\# 9 $$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)}$$

$$= \lim_{z \to z_0} \frac{\frac{f(z) - f(z_0)}{z - z_0}}{\frac{g(z) - g(z_0)}{z - z_0}} = \frac{f'(z_0)}{g'(z_0)}$$

\# 11 $\forall n$ Can find $z_n \to z_0$

$f(z_n) = c$

if z_0 is an ~~isolated~~ removable singularity

then $f(z) = c$ $\rightarrow\leftarrow$

if z_0 is a pole of order k

then $f(z) = \frac{g(z)}{(z - z_0)^k}$ $g(z_0) \neq 0$, g analytic

$$f(z) \cdot (z - z_0)^k = g(z)$$

$$g(z_n) = c\,(z_n - z_0)^k \to 0$$

$\downarrow$

$g(z_0)$

#12 $g(z) = f(\frac{1}{z})$ g is analytic when (9)

when $z \neq 0$ $\quad |\frac{1}{z}| > r$ i.e. $\frac{1}{r} > |z|$

$$\lim_{z \to \infty} f(z) = \lim_{z \to \infty} g(\frac{1}{z}) := g(0)$$

So $g(z) = \begin{cases} f(\frac{1}{z}) & 0 < |z| < \frac{1}{r} \\ 0 & z = 0 \end{cases}$

is analytic on $D_{\frac{1}{r}}(z)$

$$g(z) = \sum_{1}^{\infty} a_n z^n$$

$$f(z) = g(\frac{1}{z}) = \sum_{1}^{\infty} a_n z^{-n}$$

pole of order 1

#13 $f(z)$ has singularities at $z = 0, \pm 1$

$$f(z) = \frac{1}{z(z-1)(z+1)} = \frac{1}{z}(1 + z^2 + z^4 + z^6 + \cdots)$$

#14 $f(z) = \frac{1}{z} - \frac{1}{3!}\cdot\frac{1}{z^3} + \frac{1}{5!}\cdot\frac{1}{z^5} - \frac{1}{7!}\cdot\frac{1}{z^7}$

essential (Similar to # 3.4.12)

#15 $f(z) = \frac{1}{2}z^2 + \frac{1}{3!}z^3 + \cdots$

$$= \frac{1}{2} + \frac{1}{3!}z^1 + \frac{1}{4!}z^2 + \cdots$$

z^2 removable

#16 $f(z)$ has simple poles at $z = 2\pi i k$ (10).

$k \in \mathbb{Z}$ $k \neq 0$

when $z = 0$

$$\frac{1}{e^z - 1} - \frac{1}{z} = \frac{1}{z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots} - \frac{1}{z}$$

$$= \frac{1}{z\left(1 + \frac{z}{2} + \frac{z^2}{6} + \cdots\right)} - \frac{1}{z}$$

$$= \frac{1}{z}\left(1 - \frac{1}{2}z + \frac{1}{12}z^2 + \cdots\right) - \frac{1}{z}$$

$$= \frac{1}{z}\left(-\frac{1}{2}z + \frac{1}{12}z^2 + \cdots\right)$$ removable at $z = 0$

$f(z) = -\frac{1}{2}$

#17 $\log z = +(z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \frac{1}{4}(z-1)^4 + \cdots$

$$\frac{\log z}{(z-1)^2} = \frac{1}{z-1}\left(1 - \frac{1}{2}(z-1) + \frac{1}{3}(z-1)^2 - \frac{1}{4}(z-1)^3 + \cdots\right)$$

pole of order 1 at $z = 1$

3.5

#1 $z = x + iy$ $z^2 - 1 = (x^2 - y^2 - 1) + 2i\,xy$

$$|z^2 - 1|^2 = (x^2 - y^2 - 1)^2 + 4x^2y^2$$

use Lagrange multiplier w/ $x^2 + y^2 = 1$

another way

$z = e^{i\theta}$ $\quad |z^2 - 1| = |e^{2i\theta} - 1|$

$\max = 2 \quad \theta = \pm\frac{\pi}{2}$

#2 $|e^{x+iy}| = |e^x| \quad x = \cos\theta$

$x = 1$ when $\theta = 0$

#3 $|(z-1)^2| = |z-1|^2$

$|z-1|$ achieves its max at $z = 0,\ 1+i,\ 1-i$

#5 if $f \neq 0$ then $\frac{1}{f}$ is analytic

However $\max |f| = \min |f| = 1$

therefore $|f| = 1$ [so is $|\frac{1}{f}| = 1$]

f has local max $\rightarrow\leftarrow$

#6 if U contains no zeroes $\frac{1}{f}$ is analytic on U

K is actually cpt

$|\frac{1}{f}|$ achieves its max / min in K

$|\frac{1}{f}| \geq m > 0$ in K, on K^c $|\frac{1}{f}| \geq \frac{1}{c} \Rightarrow |\frac{1}{f}| \geq$ mi

$|f| \leq \frac{1}{\text{min}}$

#7 $|f(z)| \leq 1$ $f(0) = 0$ $f(z) = z^2 h(z)$ #12

$\Rightarrow |f(z)| \leq |z|$, $|f'(0)| \leq 1$

now define $g(z) = \begin{cases} \frac{f(z)}{z} & \text{when } z \neq 0 \\ f'(0) & z = 0 \end{cases}$

$|g(z)| \leq 1$

$g(0) = 0$

$\Rightarrow |g(z)| \leq |z|$

$\left|\frac{f(z)}{z}\right| \leq |z| \Rightarrow |f(z)| \leq |z|^2$

#8 $u + iv_1 = f_1$

$u + iv_2 = f_2$

$f_1 - f_2 = i(v_1 - v_2)$

$Re(f_1 - f_2) = 0 \Rightarrow f_1 - f_1 = C$

#10

$f(z) = \frac{2z-1}{z-2} \Rightarrow f(0) = \frac{1}{2}$

$f^{-1}(z) = \frac{2z-1}{z-2}$

10 $e^{f(z)} = c$

$f(z) = \alpha + i2\pi k$

however $f(z)$ is connected

$f(z)$ is const

13 $u(x, y) = e^{x} \cos y = \mathrm{Re}\, e^{x+iy}$

$v(x, y) = e^{x} \sin y$

14 $u(x, y) = \frac{1}{2} \log r^2 = \log r$

$v(x, y) = \theta = \tan^{-1} \frac{y}{x}$

15 (# 14)

$u(x, y) = \log r$ when $U = \mathbb{C} \setminus \{0\}$