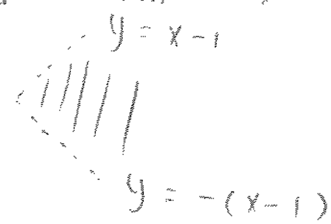
\# 1. it is path-connected $\Rightarrow$ connected

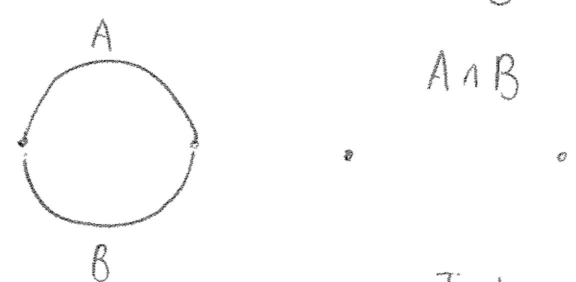
\# 2 $B_1(0,0)$ is separated from the segment.

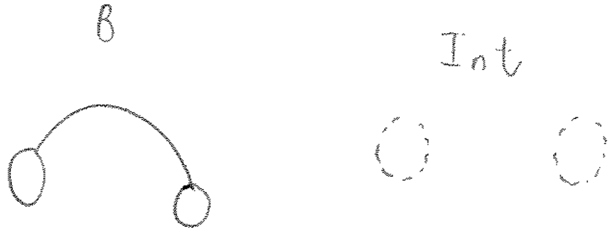
$U = B_1(0,0)$ $\quad V =$ 

\# 3 path-connected

\# 4 open & non-path connected $\Rightarrow$ not connected

\# 5 $\mathbb{R} \setminus \mathbb{Z}$ is connected

\# 8 

\# 11 

(#12) By p.f. $E \subset G$ is a component

$\forall x_n \in E$ $\quad x_n$ conv. $\quad x_n \to x \in G$

If $x \notin E$ $\quad x \in G \setminus E$

then $\exists\ U \supset E$ $\quad V \supset G \setminus E \ni x$

$U \cap V = \phi$ Howeover $B_\varepsilon(x) \subset V$

$U \supset E,$ $\ni x_{N+1}, x_{N+2}, \ldots \in B_\varepsilon(x) \subset V$

8.1

#1 $f(x,y) = \begin{cases} r\cos\theta\sin^2\theta & \text{if } r \neq 0 \\ 0 & \text{otherwise} \end{cases}$

$\lim_{r\to 0} r\cos\theta\sin^2\theta = 0$ Yes. It is cont.

#3 $f(x,0) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

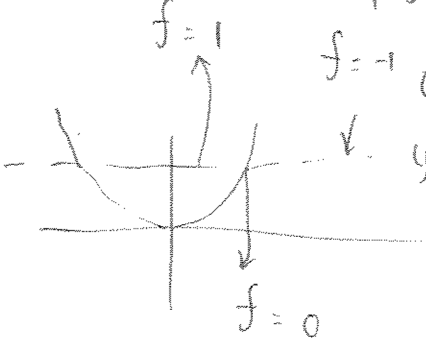
No. not cont.

#4 $f(x,y) = \begin{cases} xy & \text{if } xy \geq 0 \\ 0 & \text{if } xy < 0 \end{cases} = \begin{cases} xy & \text{if } xy > 0 \\ 0 & \text{if } xy \leq 0 \end{cases}$

$f(x,0) = 0 = f(0,y)$

Yes. It is cont.

#5 $f(x,y) = \begin{cases} \frac{(y-x^2)}{|y-x^2|}\, y = y & \text{if } y > x^2 \\ = -y & \text{if } y < x^2 \\ 0 & \text{if } y = x^2 \end{cases}$



No. not cont.

#11

Define $f(r,\theta) = \tan\frac{\pi}{2} r$ $\quad f'(r) = \left(\sec^2\frac{\pi}{2} r\right)\frac{\pi}{2}$

Cauchy $\not\Rightarrow$ Cauchy

Solution Manual

9.1 #1

$$f(x,y) = (x^2+y^2)^{+\frac{1}{2}}$$

$$f_x = \frac{\frac{1}{2}\cdot 2x}{\sqrt{x^2+y^2}^{\,3}} = \frac{x}{\sqrt{x^2+y^2}^{\,3}}$$

f_x doest not exist at $(0,0)$

$$f_y = \frac{y}{\sqrt{x^2+y^2}^{\,3}}$$

#3 $f_x = \cos y \quad f_y = -x \sin y$

$$f_{xy} = f_{yx} = -\sin y$$

#6
$$\frac{\partial f}{\partial x} = \frac{2x\,(x^2+y^2)^p - p\,(x^2+y^2)^{p-1}\,2x^3}{(x^2+y^2)^{2p}}$$

$$= \frac{2x\,(x^2+y^2) - 2p\,x^3}{(x^2+y^2)^{p+1}}$$

$$= \frac{1}{r^{2p-1}}\left(2\cos\theta - 2p\cos^3\theta\right) = 0 \quad \text{if } 2p-1$$

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x} \text{ exists when } 2p-1 < 0$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$ (2)

$$= \lim_{h\to 0} h^{1-2p} = 0 \quad \text{if} \quad 2p-1<0$$

So when $p < \frac{1}{2}$, $\frac{\partial f}{\partial x}$ is cont.

#7 just calculate

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$$

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

(9.1.5) $$f_{xy}(0,0) = \lim_{h\to 0} \frac{\frac{-h^5}{h^4} - 0}{h} = -1$$

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{\frac{h^5}{h^4} - 0}{h} = 1$$

#8 $$\lim_{r\to 0} f(x,y) = \lim_{r\to 0} \cos\theta \sin\theta \quad \text{DNE}$$

$$f_x(x,y) = \frac{y(x^2+y^2) - 2x^2y}{(x^2+y^2)^2} = \frac{\sin\theta - 2\cos^2\theta\sin\theta}{r}$$

$$f_x(0,0) = \lim \frac{f(h,0) - f(0,0)}{} = \ldots$$

$\lim_{r \to 0} f_x(x,y)$ DNE $\neq f_x(0,0)$ (3)

Symmetric in f_y

9.2 #2

$$L = \begin{bmatrix} y-2 & x+1 \\ 2x+1 & 2y-3 \end{bmatrix} \Bigg|_{(0,0)}$$

$$= \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}$$

$$F(0) + L \begin{bmatrix} x-0 \\ y-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ +1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 1-2x+y \\ 6+x-3y \end{bmatrix}$$

#3

$$L \Big|_{(1,-1)} = \begin{bmatrix} -3 & 2 \\ 3 & -5 \end{bmatrix}$$

$$F(1,-1) + \begin{bmatrix} -3 & 2 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x-1 \\ y+1 \end{bmatrix} = \begin{bmatrix} -3-3(x-1)+2(y+1) \\ 12+3(x-1)-5(y+1) \end{bmatrix}$$

#4

$$L \Big|_{(1,\pi)} = \begin{pmatrix} \frac{y}{x} & \ln x \\ e^y & x e^y \\ y \cos xy & x \cos xy \end{pmatrix}_{\substack{x=1 \\ y=\pi}} = \begin{bmatrix} \pi & 0 \\ e^{\pi} & e^{\pi} \\ -\pi & -1 \end{bmatrix}$$

$$G(x,y) \sim G(1,\pi) + \begin{bmatrix} \pi & 0 \\ e^{\pi} & e^{\pi} \\ -\pi & -1 \end{bmatrix} \begin{bmatrix} x-1 \\ y-\pi \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} 0 + \pi(x-1) \\ e^{\pi} + e^{\pi}(x-1) + e^{\pi}(y-\pi) \\ 0 - \pi(x-1) - (y-\pi) \end{bmatrix}$$

#5

$$[f_x, f_y, f_z] = \left(y^2 \cos xz - x y^2 z \sin xz,\ 2xy \cos xz - x^2 y^2 \sin xz \right) \Big|_{(1,1,\frac{\pi}{2})}$$

$$= \left(-\frac{\pi}{2}, 0, -1\right)$$

$$f(x,y,z) \sim f\left(1,1,\frac{\pi}{2}\right) + \begin{bmatrix} -\frac{\pi}{2} & 0 & -1 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \\ z-\frac{\pi}{2} \end{bmatrix}$$

$$= -\frac{\pi}{2}(x-1) - \left(z - \frac{\pi}{2}\right)$$

#6

$$\gamma'(t) = (2\pi \cos 2\pi t, -2\pi \sin 2\pi t, 2t)$$

$$\gamma(t) \sim \gamma(1) + \gamma'(1)(t-1)$$

$$= (0, 1, 1) + (2\pi, 0, 2)(t-1)$$

$$= \left(2\pi(t-1), 1, 1 + 2(t-1)\right)$$

#9

$$\frac{\partial f}{\partial x} = \frac{3x^2(x^2+y^2) - x^3 \cdot 2x}{(x^2+y^2)^2} = 3\cos^2\theta - 2\cos^4\theta$$

(5).

$$\frac{\partial f}{\partial y} = \frac{-x^3 \cdot 2y}{(x^2+y^2)^2} = -2\sin\theta\cos^3\theta$$

when $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim \frac{\frac{h^3}{h^2} - 0}{h} = 1$$

$$\frac{\partial f}{\partial y}(0,0) = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y} \quad \text{DNE}$$

$$\lim_{\substack{\cancel{\theta} \to 0 \\ r}} \frac{f(x,y) - f(0,0) - [1 \quad 0]\begin{bmatrix} x \\ y \end{bmatrix}}{\sqrt{x^2+y^2}}$$

$$= \lim_{\substack{\cancel{\theta} \to 0 \\ r}} \frac{r\cos^3\theta - r\cos\theta}{r\cancel{\theta}} = \lim_{r\to 0} \cos^3\theta - \cos\theta$$

DNE

Solution Manual

9.3

#2 $g'(t) = \frac{\partial f}{\partial x}(tx, ty)\,x + \frac{\partial f}{\partial y}(tx, ty)\,y$

$g'(1) = \frac{\partial f}{\partial x}(x, y)\,x + \frac{\partial f}{\partial y}(x, y)\,y$

#3 $f(tx, ty) = t^n f(x, y)$

$\frac{\partial f}{\partial x}(tx, ty)\,x + \frac{\partial f}{\partial y}(tx, ty)\,y = n t^{n-1} f(x, y)$

$t = 1$

$\frac{\partial f}{\partial x}(x, y)\,x + \frac{\partial f}{\partial y}(x, y)\,y = n\, f(x, y)$

#4 $g(x, y) = f(xy)$

$\frac{\partial g}{\partial x}(x, y) = f'(xy)\,y$ $\Rightarrow$ $x\frac{\partial g}{\partial x} - y\frac{\partial g}{\partial y} = 0$

$\frac{\partial g}{\partial y}(x, y) = f'(xy)\,x$

#5 $h(x, y) = f(x - y) + g(x + y)$

$h_x(x, y) = f'(x - y) + g'(x + y)$

$h_{xx}(x, y) = f''(x - y) + g''(x + y)$

$$h_y = f'(x-y)(-1) + g'(x+y) \qquad (2.)$$

$$h_{yy} = f''(x-y)(-1)^2 + g''(x+y)$$

$$\Rightarrow \quad h_{xx} - h_{yy} = 0$$

#8

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r} = \frac{\partial u}{\partial r}$$

etc.

$$\begin{array}{cccc} & x & y & z \\ \frac{\partial}{\partial r} & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{\partial}{\partial \theta} & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ \frac{\partial}{\partial \phi} & -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{array} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} \\ \frac{\partial u}{\partial \phi} \end{bmatrix}$$

$$x = r\sin\theta\cos\phi \qquad z = r\cos\theta$$

$$y = r\sin\theta\sin\phi$$

#10 $d[F_1^2(x) + F_2^2(x) + \cdots + F_8^2(x)]$

$$= 2F_1(x)\,dF_1(x) + 2F_2(x)\,dF_2(x) + \cdots + 2F_8(x)\,dF_8(x)$$

$$= \Big[2F_1(x)\frac{\partial F_1}{\partial x_1}(x) + 2F_1(x)\frac{\partial F_2}{\partial x_1}(x) + \cdots + 2F_8(x)\frac{\partial F_8}{\partial x_1}(x),$$

$$\ldots,\; 2F_1(x)\frac{\partial F_1}{\partial x_2}(x) + \cdots + 2F_8(x)\frac{\partial F_8(x)}{\partial x_n}\Big] = \vec{0}$$

Solution Manual

9.4

#1

$$df = (\sin z,\ \cos z,\ x\cos z - y\sin z)$$

$$df|_{(1,2,\frac{\pi}{4})} = \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)$$

#2

$$df = (2x+y,\ 3y^2+x)\,|_{(1,1)}$$

$$= (3,\ 4)$$

greatest ascent : $\left(\frac{3}{5},\ \frac{4}{5}\right)$

rate = 0 $\quad \left(-\frac{4}{5},\ \frac{3}{5}\right)$

#3

$$\gamma'(t) = \left(3t^2,\ -\frac{1}{t^2},\ 2e^{2t-2}\right)$$

$$\gamma'(1) = (3,\ -1,\ 2)$$

$$l(t) = (1,\ 1,\ 1) + t(3,\ -1,\ 2)$$

#4

$$\gamma(t) = (\cos t,\ \sin 2t) \quad \pi < t < 2\pi$$

$$\cos t = 0 \qquad \sin 2t = 0 \Rightarrow t = \frac{3\pi}{2}$$

$$\gamma'(t) = (-\sin t,\ 2\cos 2t)$$

$$l(t) = (0,\ 0) + t(1,\ -2)$$

#6 $g(x) = x_1^2 + x_2^2 + \cdots + x_p^2$ (2)

$dg = (2x_1, 2x_2, \ldots, 2x_p) = 2x$

#7 $f(t) = \| \gamma(t) \|^2 = \gamma(t) \cdot \gamma(t)$

$f'(t) = 2\gamma'(t) \cdot \gamma(t)$

$f''(t) = 2\gamma''(t) \cdot \gamma(t) + 2\gamma'(t) \cdot \gamma'(t)$

$f'(t_0) = 0$ since $\gamma(t_0) = 0$

$f''(t_0) = 2 \| \gamma'(t_0) \|^2 > 0$

$f(t_0)$ is a local min.

#8 $uv = 2 \quad u^2 = 4 \quad v^2 = 1 \qquad u > 0$ & $v > 0$

$\Rightarrow u = 2, \ v = 1$

$\frac{\partial G}{\partial u} = (v, 2u, 0) \qquad \frac{\partial G}{\partial v} = (u, 0, 2v)$

$P(u,v) = (2, 4, 1) + (1, 4, 0)(u-2)$

$+ (2, 0, 2)(v-1)$

#11 $x^2 + y^2 - z = 0 \qquad \text{gradient} = (2x, 2y, -1) \big|_{(1,2,5)}$

$= (2, 4, -1)$

$(2, 4, -1) \cdot (x-1, y-2, z-5) = 0$

$2(x-1) + 4(y-2) - (z-5) = 0$

#12 around $a = \sin\theta_0 \cos\varphi_0$ $\quad \sin\theta_0 = \sqrt{a^2+b^2}$ (3).

$b = \sin\theta_0 \sin\varphi_0 \Rightarrow \cos\varphi_0 = \dfrac{a}{\sqrt{a^2+b^2}}$

$c = \cos\theta_0 \quad \sin\varphi_0 = \dfrac{b}{\sqrt{a^2+b^2}}$

the surface is parametrized by θ, φ

$$G = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$|\theta - \theta_0| < \varepsilon \quad |\varphi - \varphi_0| < \varepsilon$$

Tangent plane : $(a, b, c) + \dfrac{\partial G}{\partial \theta}(\theta - \theta_0) + \dfrac{\partial G}{\partial \varphi}(\varphi - \varphi_0)$

$$\frac{\partial G}{\partial \theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$$

$$\frac{\partial G}{\partial \varphi} = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$$

$$\frac{\partial G}{\partial \theta}(\theta_0, \varphi_0) = \left(\frac{ca}{\sqrt{a^2+b^2}}, \frac{cb}{\sqrt{a^2+b^2}}, -\sqrt{a^2+b^2}\right)$$

$$\frac{\partial G}{\partial \varphi}(\theta_0, \varphi_0) = (-b, a, 0)$$

9.5

#1 $f(x, y) = [(x-1)+1]^2 + [(x-1)+1][(y-2)+2]$

$= 3 + 4(x-1) + (y-2) + (x-1)^2 + (x-1)(y-2)$

#2 $f(x, y) \sim 1 + xy + \frac{1}{2}x^2y^2$

#3 $f(x) = f(a) + df(a)(x-a) + \frac{1}{2} d^{n+1} f(c)(x-a)^2$ (4)

$\| f(x) - f(a) - df(a)(x-a) \|$

$= \| \frac{1}{2} d^2 f(c)(x-a)^2 \| = | \frac{1}{2}(x-a) \cdot d^2 f(c)(x-a)$

$\leq \frac{1}{2} \| d^2 f(c) \| \| x-a \|^2 \leq M \| x-a \|^2$

#7 $g(x) = u \cdot F(x)$

$dg = u \cdot dF = [u_1 \frac{\partial F_1}{\partial x_1} + u_2 \frac{\partial F_2}{\partial x_1} + \cdots + u_8 \frac{\partial F_8}{\partial x_1}, \cdots$

$\cdots, u_1 \frac{\partial F_1}{\partial x_p} + \cdots + u_8 \frac{\partial F_8}{\partial x_p}]$

$g(b) - g(a) = dg(c) \cdot (b-a)$

#8 $= u\, dF(c) \cdot (b-a)$

$df = [-4x - 2y, -2x + 2y] = 0$

$\Rightarrow x = y = 0$

$d^2 f = \begin{bmatrix} -4 & -2 \\ -2 & 2 \end{bmatrix}$ $\Delta = -12 < 0$

$(0,0)$ saddle

#9 $df = [2x - 2y, \ 3y^2 + 2y - 2x - 3] = 0$ (J)

$\Rightarrow x = y \qquad y = \pm 1$

$(1,1)$ or $(-1,-1)$

$$d^2 f = \begin{bmatrix} 2 & -2 \\ -2 & 6y+2 \end{bmatrix}$$

$\Delta = 12 > 0 \qquad f_{xx} = 2 > 0 \qquad (1,1)$ local min

$\Delta = -12 < 0$ at $(-1,-1) \Rightarrow$ Saddle

9.6

#1 $f(x) = \sin x$

$f'(x) = \cos x$

when $\cos x \neq 0$ i.e. $x \neq \frac{2k+1}{2}\pi \quad k \in \mathbb{Z}$

f has a smooth local inverse

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-y^2}}$$

#2 $dF = \begin{bmatrix} 2x & 2y \\ 1 & -1 \end{bmatrix} \qquad \Delta = -2(x+y) \neq 0$

when $x \neq -y$

$$(dF)^{-1} = \frac{1}{2(x+y)} \begin{bmatrix} -1 & -2y \\ -1 & 2x \end{bmatrix}$$

$F(x,y) = (u,v)$ $\quad u = x^2 + y^2$

$v = x - y$

$v^2 = x^2 + y^2 - 2xy$

$$v^2 - u = -2xy$$

$$u + u - v^2 = (x+y)^2$$

$$\pm\sqrt{2u - v^2} = x + y$$

$$v = x - y$$

$\Rightarrow$ calculate x, y interm of u, v

#3 ~~dF = ... sinθ ... sinθ ... tanθ sin~~

$$dF = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \rho\cos\theta\cos\varphi & \rho\cos\theta\sin\varphi & -\sin\theta \\ -\rho\sin\theta\sin\varphi & \rho\sin\theta\cos\varphi & 0 \end{pmatrix}$$

$$\Delta = +\rho\sin^3\theta\sin^2\varphi + \rho\sin\theta\cos^2\theta\cos^2\varphi$$

$$+\rho\sin\theta\cos^2\theta\sin^2\varphi + \rho\sin^3\theta\cos^2\varphi$$

$\Delta = \rho \sin^3 \theta + \rho \sin\theta \cos^2\theta$ (7).

$= \rho \sin\theta$

$\Delta = 0 \Rightarrow \rho = 0$ or $\theta = 0$ or $\theta = \pi$

the derivative of the inverse exists

I really hate to find the inverse matrix ü

#4

$x = u^4 - u + ux + v^2$

$y = \cos u + \sin v$

$$dF = \begin{pmatrix} 4u^3 - 1 - v & u + 2v \\ -\sin u & \cos v \end{pmatrix} \Bigg|_{(0,0)}$$

$\Delta = -1 \neq 0$

$$dF^{-1} = \frac{1}{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

#5

$F(r, \theta) = (r\cos\theta, r\sin\theta) = (x, y)$

$$dF = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \Bigg|_{(1, \frac{\pi}{2})}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$(dF)^{-1} = \frac{1}{1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

the inverse fcn is given by $r = \sqrt{x^2 + y^2}$, $\theta = \cos^{-1} \frac{x}{r}$

\# 6 $dF = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (8)

the inverse fcn is given by

$r = \sqrt{x^2 + y^2}$

$\theta = \sin^{-1} \dfrac{y}{\sqrt{x^2+y^2}} + 2\pi$

\# 7 by contradiction

assume $a \neq b$ yet $F(a) = F(b)$

$\phi(t) = u \cdot F(a + tu)$

where $u = \dfrac{b-a}{\|b-a\|}$ $t \in [0, \|b-a\|]$

$\phi'(c) = 0$ according to the MVT

i.e. $u^\theta \cdot dF(c)\, u = 0$ $\rightarrow\leftarrow$

Solution Manual

#1 $\frac{\partial f}{\partial y} = x \cdot 6y + 6y^2 = 0$ (1)

when $x = -y$ or $y = 0$

$\downarrow$ $\downarrow$

$\left(\frac{1}{\sqrt[3]{2}}, -\frac{1}{\sqrt[3]{2}} \right)$ $(1, 0)$

#2 $\frac{\partial f}{\partial z} = x + y + \cos(x+y+z) \Big|_{(0,0,0)} = 1 \neq 0$

Ans: Yes

#3 $\begin{bmatrix} 2u & 2v \\ x & y \end{bmatrix}$

$\Delta \neq 0$ when $uy \neq vx$

#4 $\begin{bmatrix} 2u+2 & 2v \\ 3u^2 - x & \cos v + y \end{bmatrix}$

$\Delta = 2 \neq 0$

#5 $\begin{bmatrix} 3u^2 & 2x^2 v & 1 \\ 2y^2 u & 3v^2 & 1 \\ 0 & 0 & 2w + x \end{bmatrix}$

$\Delta = 5 \neq 0$

#6 $df = \begin{bmatrix} y+z \\ x+z \\ x+y \end{bmatrix}$

rank = 1 $\Rightarrow$ $f(x,y,z) = 1$ is a smooth 2-surface

if only if not

$x = -y$

$y = -z$

$z = -x$

i.e. $(0,0,0)$

Hower $(0,0,0) \notin S$

tangent plane: $(b+c)(x-a) + (a+c)(y-b) + (a+b)(z-c) = 0$

i.e. $(b+c)x + (a+c)y + (a+b)z = 2$

#7 $dF = \begin{bmatrix} 2x & 2y & -2z \\ 1 & 1 & 1 \end{bmatrix}$

rank = 2 if $(x,y,z) \neq (0,0,0)$

tangent vector $= \begin{vmatrix} i & j & k \\ 2x & 2y & -2z \\ 1 & 1 & 1 \end{vmatrix}$

$= [2(y+z), -2(x+z), 2(x-y)]$

Another way:

$$x^2+y^2=z^2$$

$$x+y=-z \Rightarrow x^2+y^2+2xy=z^2$$

$\Rightarrow 2xy=0$ either $x=0$ or $y=0$

when $x=0$ $y=-z$ & $y^2-z^2=0$

$(0,-z,z) = S =$ line

when $y=0$ $(-z,0,z) = S =$ line

#8

$$\begin{bmatrix} 2x & 2y & 2u & -3 \\ 2+y & x-1 & 6u & -9 \end{bmatrix} = dF$$

If rank $=2 \Rightarrow S$ is a smooth 2 surface

iff only if not $2x = 3(2+y)$

$2y = 3(x-1)$

$$(x,y) = \left(-\frac{3}{5}, \frac{12}{5}\right)$$

Cannot solve u, v

#9

$$\begin{bmatrix} e^u & e^u & xe^u+ye^u & 0 \\ v & u & y & x \end{bmatrix} = dF$$

If rank $=2 \Rightarrow S$ is a smooth 2-surface

If not, $u=v$, $x(x+y)=0$, $y=v(x+v)$

Solution Manual

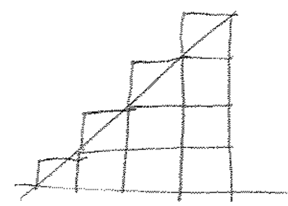
10.1 #1 $U(f,P) = \sum_{i,j=1}^{4} x_i y_j \Delta x \Delta y = \frac{100}{16} \cdot \frac{1}{16}$

$x_i = \frac{i}{4} \qquad y_j = \frac{j}{4} \qquad \Delta x \Delta y = \frac{1}{4}$

$L(f,P) = \sum_{i,j=1}^{4} x_{i-1} y_{j-1} \Delta x \Delta y = \frac{36}{16} \cdot \frac{1}{16}$

#2 $U(\chi_\Delta, P) = 10/16$

$L(\chi_\Delta, P) = 6/10$



#3

$$\lim_{n\to\infty} U(f, P_n) - L(f, P_n)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\sup_{I_i} f - \inf_{I_i} f \right) \Delta x_n$$

$$\sup_{I_i} f \geq f(x_0) > \sup_{I_i} f - \varepsilon \qquad x_0, x_1 \in I_i$$

$$\inf_{I_i} f \leq f(x_1) < \inf_{I_i} f + \varepsilon$$

$$\leq \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_0) - f(x_1)] \Delta x_n + 2\varepsilon \, \mathrm{vol}(R)$$

$$\leq K \lim_{n\to\infty} \sum_{i=1}^{n} |g(x_0) - g(x_1)| \Delta x_n + 2\varepsilon \, \mathrm{vol}(R)$$

$$\leq K \lim_{n\to\infty} \sum_{i=1}^{n} \left(\sup_{I_i} g - \inf_{I_i} g \right) \Delta x_n + 2\varepsilon \, \mathrm{vol}(R)$$

#7 $\int_R K \, dV(x)$

$= U(K, P) = L(K, P) = K \operatorname{vol}(R)$

#8
$-M \leq f(x) \leq M$ on R

$$\int_R f(x) \, dV(x) = \lim_{n \to \infty} \sum_{i=1}^{n} \sup_{I_i} f \, \Delta x_n$$

$$\leq M \operatorname{vol}(R)$$

Similarly, $\int = \lim \sum_{i=1}^{n} \inf_{I_i} f \, \Delta x_n$

$$\geq -M \operatorname{vol}(R)$$

$\Rightarrow \left| \int_R f(x) \, dV(x) \right| \leq M \operatorname{vol}(R)$

#9 If f is cont. on R, cpt
then f is uniformly cont.
$\forall \varepsilon \; \exists \delta$ s.t. $\forall \; |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Choose a partition so that any two points in the same section has distance less than δ

$$\lim_{n \to \infty} \sum_{i=1}^{n} \left(\max_{I_i} f - \min_{I_i} f \right) \Delta x_n$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_0) - f(x_1)] \Delta x_n < \varepsilon \operatorname{vol}(R)$$

#10

$$\chi_{A\cap B} = \chi_A \chi_B$$

$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \chi_B$$

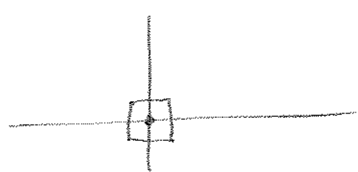
$$B \subset A \Rightarrow \chi_B = \chi_A \chi_B$$

$$\chi_{A\setminus B} = \chi_A - \chi_B$$

10.2

#3 Just need to prove $vol(\{a \text{ point}\}) = 0$

Let's assume it is the origin.



$$\underline{V}(E) = 0$$

$$\overline{V}(E) = \lim_{n\to\infty} \frac{1}{n^2} = 0$$

#4 $\underline{V}(E) = 0 \neq \overline{V}(E) = 1$

No. not a Jordan Region.

#5 $v(A\cup B) = \underset{0}{\underset{\parallel}{v(A)}} + \underset{0}{\underset{\parallel}{v(B)}} - v(A\cap B)$

Can it be negative ü

#6 $df \neq 0 \Rightarrow$ either $f_x \neq 0 \Rightarrow x = f(y)$

or $f_y \neq 0 \Rightarrow y = g(x)$

f is smooth

Can cut K into union of E_i

E_i : either $(x, f(x))$

or $(g(y), y)$

Use # 10.2.11

\# 7 Use # 6. ∂E of vol zero.

Thm. 10.2.9. $E \Rightarrow$ Jordan.

10.3

\# 7 $$\int_B f(x)\,dV(x) = \int_A f(x)\chi_B(x)\,dV(x)$$

$$\leq \int_A f(x)\chi_A(x)\,dV(x) = \int_A f(x)\,dV(x)$$

\# 9 $|f| \leq M$ $\quad \mathrm{vol}(A) = 0$

$$\left|\int_A f\,dV\right| \leq M\,\mathrm{vol}(A) = 0$$

$$\int_A f\,dV = 0$$

\# 10 the best way to understand the problem

is try to think $\frac{1}{V(A)}\int f(x)\,dV(x)$

as ~~weight~~ a weighted sum of f

$\text{avg}(f, A) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\omega_i \qquad \Sigma\,\omega_i = 1$ (5)

Since f is cont. on A connected

So $\exists\, x_n$ s.t. $f(x_n) = \sum_{i=1}^{n} f(x_i)\,\omega_i$

Image (f) is cpt

$x_{n_j} \to x_0 \qquad \lim f(x_{n_j}) = f(x_0)$

#12

$$g(x, y) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sin kx \; \sin ky$$

$$|g(x, y)| \leq \Sigma \frac{1}{k^2} < \infty$$

By Weierstrass Thm, get uniform Conv.

$$\int \sum_{k=1}^{\infty} \frac{1}{k^2} \sin kx \; \sin ky = \sum_{k=1}^{\infty} \frac{1}{k^2} \int \sin kx \; \sin ky \; dx\,dy$$

nice fcn

$\Rightarrow$ integrable

$$\int \sin kx \, dx = \frac{1}{k}(-\cos kx)\Big|_a^b \leq \frac{2}{k}$$

$$\int \sin ky \, dy \leq \frac{2}{k}$$

Solution Manual

10.4 1-8, 11-12

#1 $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sin kx \, dx \, \sin ky \, dy$

$= \left(-\frac{1}{k}\cos kx \Big|_{-\pi}^{\pi}\right)\left(-\frac{1}{k}\cos ky \Big|_{-\pi}^{\pi}\right)$

$= 0$

By the uniform conv. $\underbrace{\int_{x,y \in [-\pi,\pi]\times[-\pi,\pi]} g(x,y)\,dx\,dy}_{= 0}$

#2 $\frac{y^3 x}{(1+y^2x^2)^2}$ is cont. on $[0,1]\times[0,1]$

$\Rightarrow$ integrable

$\int_{y=0}^{1}\int_{x=0}^{1} \frac{y^3 x}{(1+y^2x^2)^2}\,dx\,dy$ $\quad u = 1+y^2x^2$

$\quad du = 2y^2 x\,dx$

$= \int_{y=0}^{1} \frac{y}{2}\int_{1}^{1+y^2} u^{-2}\,du\,dy$

$= \int_{y=0}^{1} \frac{y}{2}\left(-u^{-1}\right)\Big|_{1}^{1+y^2}$

$= \int_{y=0}^{1} \frac{y}{2}\left(1-\frac{1}{1+y^2}\right)dy$

$= \frac{1}{4}y^2 - \frac{1}{4}\ln(1+y^2)\Big|_0^1 = \frac{1}{4} - \frac{1}{4}\ln 2$

#3

Δ at (0,0), (1,0), (1,1)

area = $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$

Δ at (0,0), (a,0), (a,b)

area = $\frac{1}{2} a b$

#4

$$\int_{x=-1}^{1} \left(\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \right) dx$$

$$= \int_{x=-1}^{1} 2\sqrt{1-x^2}\, dx \qquad x = \sin\theta$$

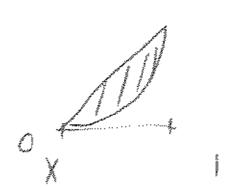
$$dx = \cos\theta\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2\theta\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2\theta + 1\, d\theta = \frac{1}{2}\sin 2\theta + \theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi$$

Also try $\int_{r=0}^{1} \left(\int_{\theta=0}^{2\pi} d\theta \right) r\, dr$

#5 Region:



integrand: $r^2 = x^2 + y^2$

in physics: it is called moment of inertia

$$\int_0^1 \left(\int_{x^2}^{x} x^2 + y^2 \, dy \right) dx$$ (3)

$$= \int_0^1 x^2 y + \frac{1}{3} y^3 \Big|_{x^2}^{x} \, dx$$

$$= \int_0^1 x^2 (x - x^2) + \frac{1}{3} (x^3 - x^6) \, dx$$

$$= \frac{1}{4} x^4 - \frac{1}{5} x^5 + \frac{1}{12} x^4 - \frac{1}{21} x^7 \Big|_0^1 = \frac{1}{4} - \frac{1}{5} + \frac{1}{12} - \frac{1}{21}$$

\# 6 $$\int_{x=-1}^{1} \left(\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\int_{z=-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz \right) dy \right) dx$$

$$= \int_{r=0}^{1} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, d\phi \, d\theta \, d\phi$$

\# 7 $$\int_{x=0}^{1} x \left(\int_{y=0}^{x^2} \left(\int_{z=0}^{x+y} dz \right) dy \right) dx$$

$$= \int_{x=0}^{1} x \int_{y=0}^{x^2} x + y \, dy \, dx$$

$$= \int_{x=0}^{1} x \quad xy + \frac{1}{2} y^2 \Big|_0^{x^2} \, dx$$

$$= \int_{x=0}^{1} x \cdot x^3 + x \cdot \frac{1}{2} x^4 \, dx$$

$$= \frac{1}{5} + \frac{1}{12}$$

#8 Omit it. Sure. How could it be wrong? (4)

#11 Need to prove $\forall\ \varepsilon\ \exists\ \delta\ \forall\ |x-x_0|<\delta$

$$\left| \int_A f(x,y)\,dv(y) - \int_A f(x_0,y)\,dv(y) \right| < \varepsilon$$

$$\leq \int_A |f(x,y) - f(x_0,y)|\,dv(y) \leq \varepsilon\, v(A)$$

by the uniform continuity on $B \times A$

#12

$$\frac{d}{dt}\int_A f(t,x)\,dv(x)$$

$$= \lim_{h\to 0} \int_A \frac{f(t+h,x) - f(t,x)}{h}\,dv(x)$$

$$= \lim_{h\to 0} \int_A \frac{\partial f}{\partial t}(t_0,x)\,dv(x) \qquad t_0 \in [t-h, t+h]$$

$$= \lim_{h\to 0} \int_A \frac{\partial f}{\partial t}(t,x)\,dv(x) = \int_A \frac{\partial f}{\partial t}(t,x)\,dv(x)$$

$$+ \lim_{h\to 0} \underbrace{\int_A \left[\frac{\partial f}{\partial t}(t_0,x) - \frac{\partial f}{\partial t}(t,x)\right] dv(x)}_{=\,0}$$

By uniform continuity

Solution Manual 10. # 3-4, 7-10 (1)

3

$$\phi(s,t) = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

if $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

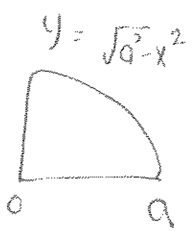
$$\int_A f(x,y)\, dv(x,y) = \int_{[0,1]\times[0,1]} f(x,y)\, |u_1 v_2 - v_1 u_2| \, ds\, dt$$

where $x = u_1 s + v_1 t$, $y = u_2 s + v_2 t$

4. Area $= |u_1 v_2 - v_1 u_2| = \left| \begin{matrix} i & j & k \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{matrix} \right|$

$$= |\vec{u} \times \vec{v}|$$

7



$$\int_{r=0}^{a} \int_{\theta=0}^{\frac{\pi}{2}} e^{r^2} r \, dr\, d\theta$$

$$= \frac{1}{2} e^{r^2} \Big|_0^a \cdot \frac{\pi}{2}$$

$$= \frac{1}{2}(e^{a^2} - 1) \cdot \frac{\pi}{2}$$

8 $u = x^2 + y^2$

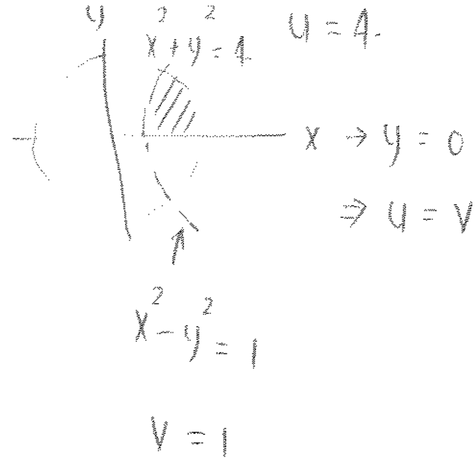
$v = x^2 - y^2$

$$du\,dv = \left| \begin{matrix} 2x & 2y \\ 2x & -2y \end{matrix} \right| dx\, dy$$

$$= |-8xy|\, dx\, dy$$

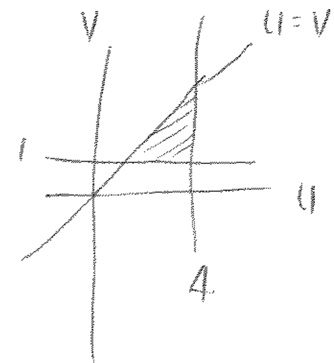
(2)

$$\int_{\substack{u\le 4,\ v\ge 1\\ u=v}} \frac{1}{u}\,\frac{1}{8}\,du\,dv$$



$$= \int_{v=1}^{4}\left(\int_{u=v}^{4}\frac{1}{8}\,\frac{1}{u}\,du\right)dv$$

$$= \int_{v=1}^{4}\frac{1}{8}\ln|u|\Big|_{v}^{4} = \int_{v=1}^{4}\frac{1}{8}\left(\ln 4-\ln v\right)dv$$



$$= \frac{1}{8}(\ln 4)\,v - \frac{1}{8}\left(v\ln v - v\right)\Big|_{1}^{4}$$

#9 $$\int_{\rho=0}^{r}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\rho^2\sin\theta\,d\rho\,d\theta\,d\phi$$

$$= \frac{1}{3}\rho^3\Big|_0^r\left(-\cos\theta\right)\Big|_0^{\pi}\cdot 2\pi = \frac{4}{3}\pi r^3$$

#10 $$\int_{z=0}^{h}\int_{r=0}^{\frac{a}{h}z}\int_{\theta=0}^{2\pi} r\,dr\,d\theta\,dz$$

$$= \int_{0}^{h}\frac{1}{2}r^2\Big|_0^{(\frac{a}{h})z}\cdot 2\pi\,dz$$

$$= \int_{0}^{h}\frac{1}{2}\left(\frac{a}{h}\right)^2 z^2\cdot 2\pi = \pi\left(\frac{a}{h}\right)^2\frac{1}{3}h^3$$