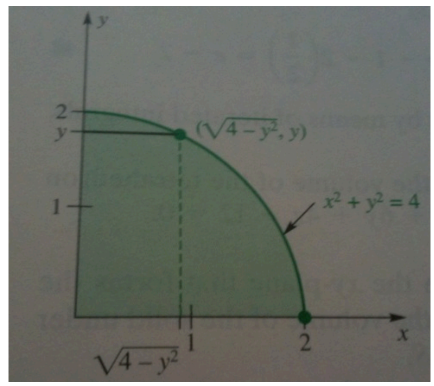
Mock Exam 3 Solutions
Problem 1

Ch 13.3, Ex. 4

Find the volume of the solid in the first octant $(x \geq 0,\ y \geq 0,\ z \geq 0)$ bounded by the circular paraboloid $z = x^2 + y^2$, the cylinder $x^2 + y^2 = 4$, and the coordinate planes.

Solution



The region S in the first quadrant of $xy-$plane is bounded by a quarter of the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = 0$. Although S can be thought of as either a y-simple or an x-simple region, we shall treat S as the latter and write its boundary curves as $x = \sqrt{4 - y^2}$, $x = 0$, and $y = 0$. Thus,

$$S = \left\{ (x, y) : 0 \leq x \leq \sqrt{4 - y^2},\ 0 \leq y \leq 2 \right\}$$

Figure 14 shows the region S in the $xy-$plane. Now our goal is to calculate $V = \iint\limits_S \left(x^2 + y^2\right) dA$ by means of an iterated integral. This time we first fix y and integrate along a line (Figure 14) from $x = 0$ and $x = \sqrt{4 - y^2}$ and then integrate the results from $y = 0$ to $y = 2$.

$$V = \iint\limits_S \left(x^2 + y^2\right) dA = \int_0^2 \int_0^{\sqrt{4-y^2}} \left(x^2 + y^2\right) dx\, dy$$

$$= \int_0^2 \left[\frac{1}{3}\left(4 - y^2\right)^{\frac{3}{2}} + y^2 \sqrt{4 - y^2} \right] dy$$

By the trigonometric substitution $y = 2\sin\theta$, the latter integral can be rewritten as

$$\int_0^{\frac{\pi}{2}} \left[\frac{8}{3}\cos^3\theta + 8\sin^2\theta + \theta\cos\theta \right] 2\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{16}{3}\cos^4\theta + 16\sin^2\theta\cos^2\theta \right] d\theta$$

$$= \frac{16}{3} \int_0^{\frac{\pi}{2}} \cos^2\theta \left(1 - \sin^2\theta + 3\sin^2\theta\right) d\theta$$

$$= \frac{16}{3} \int_0^{\frac{\pi}{2}} \left(\cos^2\theta + 2\sin^2\theta\cos^2\theta\right) d\theta$$

$$= \frac{16}{3} \int_0^{\frac{\pi}{2}} \left(\cos^2\theta + \frac{1}{2}\sin^2 2\theta \right) d\theta$$

$$= \frac{16}{3} \int_0^{\frac{\pi}{2}} \left(\frac{1 + \cos 2\theta}{2} + \frac{1 - \cos 4\theta}{4} \right) d\theta = 2\pi$$

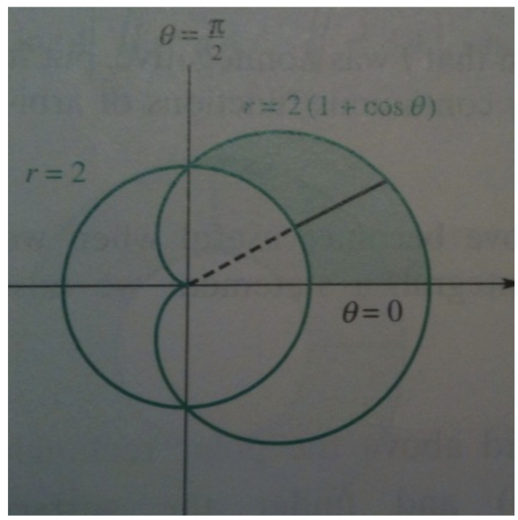
Mock Exam 3 Solutions
Problem 2

Ch 13.4, Ex 2

Evaluate $\int_S y\,dA$ where S is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1+\cos\theta)$. The graph of the carioid will be given.

Solution

Since S is an r-simple set, we write the given integral as an iterated polar integral, with r as the inner variable of integration. In this inner integration, θ is held fixed; the integration is along the heavy line of Figure 7 from $r = 2$ to $r = 2(1+\cos\theta)$.

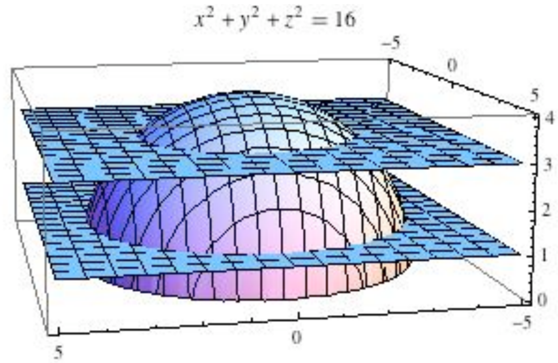


$$
\begin{aligned}
\iint_S y\,dA &= \int_0^{\frac{\pi}{2}}\int_2^{2(1+\cos\theta)} (r\sin\theta)\,r\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}}\left[\frac{r^3}{3}\sin\theta\right]_2^{2(1+\cos\theta)} d\theta \\
&= \int_0^{\frac{\pi}{2}}\left[\frac{\left(2(1+\cos\theta)\right)^3}{3}\sin\theta - \frac{2^3}{3}\sin\theta\right]d\theta \\
&= \frac{8}{3}\int_0^{\frac{\pi}{2}}\left[(1+\cos\theta)^3\sin\theta - \sin\theta\right]d\theta \\
&= \frac{8}{3}\left[\frac{-1}{4}(1+\cos\theta)^4 + \cos\theta\right]_0^{\frac{\pi}{2}} \\
&= \frac{8}{3}\left[\frac{-1}{4} + 0 - (-4+1)\right] = \boxed{\frac{22}{3}}
\end{aligned}
$$

Mock Exam 3 Solutions
Problem 3

Ch 13.6, Ex 3

Find the area of the surface G cut from the hemisphere $x^2+y^2+z^2=4^2$, $z\ge 0$, by the plane $z=1$ and $z=3$.



Solution

The surface of the hemisphere is define by $z=\sqrt{16-x^2-y^2}$.

$$z_x=\frac{-2x}{2\sqrt{16-x^2-y^2}}$$

$$z_y=\frac{-2y}{2\sqrt{16-x^2-y^2}}$$

The surface area of the hemisphere between $z=1$ and $z=3$ is

$$\iint_S \sqrt{1+(z_x)^2+(z_y)^2}$$

$$\iint_S \sqrt{1+\left(\frac{x}{\sqrt{16-x^2-y^2}}\right)^2+\left(\frac{y}{\sqrt{16-x^2-y^2}}\right)^2}$$

$$\iint_S \sqrt{\frac{16-x^2-y^2}{16-x^2-y^2}+\frac{x^2}{16-x^2-y^2}+\frac{y^2}{16-x^2-y^2}}$$

$$\iint_S \sqrt{\frac{16-x^2-y^2+x^2+y^2}{16-x^2-y^2}}$$

We will use polar coordinates to find the radius values we find the radius of the two circles created by the intersections of $z=1$ and $z=3$ with the hemisphere. We find them to be $\sqrt{7}$ and $\sqrt{15}$, respectively.

$$\boxed{\int_{\theta=0}^{2\pi}\int_{r=\sqrt{7}}^{\sqrt{15}}\sqrt{\frac{16}{16-r^2}}\,r\,dr\,d\theta=16\pi}$$

Mock Exam 3 Solutions
Problem 4

Evaluate the triple integral of $f(x,y,z)=2xyz$ over the solid region S in the first octant bounded by the parabolic cylinder $z=2-\frac{1}{2}x^2$ and the plane $z=0,\ y=x$, and $y=0$.

Solution

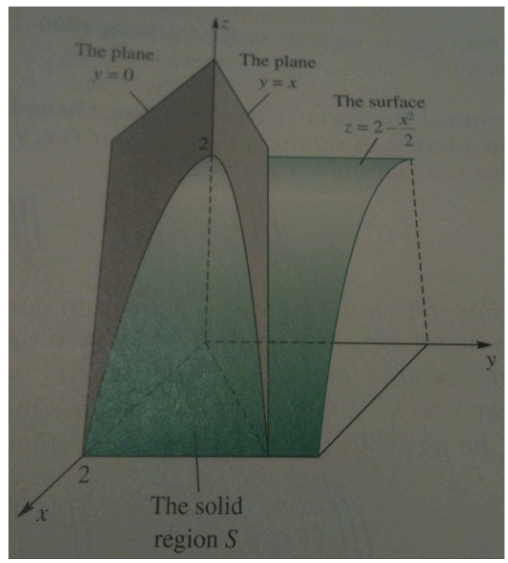
The solid region S is shown in Figure 4. The triple integral $\iiint\limits_S 2xyz\,dV$ can be evaluated by an iterated integral. We integrate along a vertical line from $z=0$ to $z=2-\frac{1}{2}x^2$.

$$\iiint\limits_S 2xyz\,dV=\int_{x=0}^{2}\int_{y=0}^{x}\int_{z=0}^{2-\frac{1}{2}x^2}(2xyz)\,dz\,dy\,dx$$

$$\int_{x=0}^{2}\int_{y=0}^{x}\left[xyz^2\right]_{z=0}^{2-\frac{1}{2}x^2}dy\,dx$$

$$\int_{x=0}^{2}\int_{y=0}^{x}\left(4xy-2x^3y+\frac{1}{4}x^5y\right)dy\,dx$$

$$\int_{x=0}^{2}\left(2x^3-x^5+\frac{1}{8}x^7\right)dx=\boxed{\frac{4}{3}}$$



Find the volume of the solid S bounded above by the sphere $\rho = 4$ and below by the cone $\phi = \frac{\pi}{3}$.

Solution

The volume is given by $V = \int_{\phi=0}^{\frac{\pi}{3}} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{4} \rho^2 \sin\phi$.

$$\int_{\phi=0}^{\frac{\pi}{3}} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{4} \left(\rho^2 \sin\phi\right) d\rho\, d\theta\, d\phi$$

$$\int_{\phi=0}^{\frac{\pi}{3}} \int_{\theta=0}^{2\pi} \tfrac{1}{3}\rho^3 \sin\phi \Big|_{\rho=0}^{4} d\theta d\phi$$

$$\int_{\phi=0}^{\frac{\pi}{3}} \int_{\theta=0}^{2\pi} \frac{64\sin\phi}{3}\, d\theta d\phi$$

$$\int_{\phi=0}^{\frac{\pi}{3}} \frac{128\pi\sin\phi}{3} d\phi = \boxed{\frac{64\pi}{3}}$$