Last time: More on Galois representations from Tate modules of elliptic curves $/K$ (good reduction, bad reduction).

Recalled Weil pairing $T_\ell E \times T_\ell E \to \mathbb{Z}_\ell(1)$

$$\mathbb{Z}_\ell(1) = \varprojlim \mu_{\ell^n}(\bar{K}) = T_\ell \mathbb{G}_m$$

$\mathbb{G}_m \cong \operatorname{Spec} K[x, x^{-1}]$ ($\bar{K}^\times$ as algebraic variety).

Today Take E an elliptic curve over $\mathbb{C}_p$ and produce a 1-dim'l subspace of

$V_p E \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ ← 2 dim'l $\mathbb{C}_p$-vector space.

(sort of Like Hodge filtration on $H_1(E(\mathbb{C}), \mathbb{Q})$.

If $E = \mathbb{C}/\Lambda$

$\Lambda \otimes \mathbb{Q}$ ← 2-dim'l $\mathbb{Q}$-vector space

$$\Lambda \otimes \mathbb{C} \twoheadrightarrow \mathbb{C}$$

2-dim'l, 1-dim'l

Kernel = Hodge Filtration).

We'll produce a non-zero map (of $\mathbb{Z}_p$-modules) $T_p E \to \mathbb{C}_p$

$$\Rightarrow \quad T_p E \otimes \mathbb{C}_p \to \mathbb{C}_p$$

2-dim'l, 1-dim'l

Kernel = 1-dim'l Hodge-Tate Filtration.

Going to use Weil pairing.

Recall: If R is a ring and $S = R[x]/f(x)$ then $\Omega_{S/R} = S/f'(x)$ (S-module).

(Recall (~) we used this as a measure of ramification
K fin. ext. $\mathbb{Q}_p$, L/K totally ramified finite extension
$\mathcal{O}_K \hookrightarrow \mathcal{O}_L$ $\quad \pi$ uniformizer in $\mathcal{O}_L$
$\mathcal{O}_L = \mathcal{O}_K[x]/F_\pi(x)$. $\quad (x \leftrightarrow \pi)$.
$\Omega_{\mathcal{O}_L/\mathcal{O}_K} = \mathcal{O}_L/F_\pi'(\pi)$.

e.g. $K = \mathbb{Q}_p$ $\quad L = \mathbb{Q}_p(\zeta_p)$

$\pi = \zeta_p - 1$. $\qquad |\pi| = |p|^{p-1}$

$$F_\pi(x) = F_{\zeta_p - 1}(x+1).$$

$$= \frac{(x+1)^p - 1}{(x+1) - 1}$$

$$= \frac{x^p + px^{p-1} + \binom{p}{2} \cdots + px}{x}$$

$$= x^{p-1} + px^{p-1} + \cdots + p.$$

Forgot to take f_π'!

$(p-1)x^{p-2} + p(\cdots)$

$|\pi^{p-2}| = |p|^{\frac{p-2}{p-1}}$.

$\Omega_{\mathcal{O}_L/\mathcal{O}_K} = \mathcal{O}_L / m_{\mathcal{O}_L}^{p-2}$

~~π~~

~~$= \pi^{p-1} + \cdots + p.$~~

~~have to compare these, have same absolute value.~~

$\mathbb{F}_p[x]/x^p$

Exercise I didn't give the exercise I meant to give because $\mathbb{F}_p[x]/(x-1)^p$ ... $\mathbb{F}_p[x]/f(x)$

Compute Ω_S for $R=\mathbb{C}$, $S = \mathbb{C}[x]/x^n$, $S=\mathbb{C}[x]$; $\mathbb{C}[x]/x^n-1$

for $R=\mathbb{F}_p$, $S=\mathbb{F}_p[x]/(x-1)^n$; $S=\mathbb{F}_p[x]$

$(p,n)=1 \Leftarrow p \mid n$.

Answer: $\Omega_{\mathbb{C}[x]/\mathbb{C}} = \mathbb{C}[x]dx$ $\qquad \Omega_{\mathbb{F}_p[x]/\mathbb{F}_p} = \mathbb{F}_p[x]dx.$

$$\Omega_{(\mathbb{C}[x]/x^n)/\mathbb{C}} \cong \mathbb{C}[x]/x^{n-1}$$

$\mathbb{C}[x]/x^n$

$\Omega_{(\mathbb{F}_p[x]/x^n)/\mathbb{F}_p}$

$(p,n)\neq 1$. $\quad \mathbb{F}_p[x]/x^{n-1}$

$p \mid n \rightsquigarrow \mathbb{F}_p[x]/x^n$

$$\Omega_{(\mathbb{F}_p[x]/x^p)/\mathbb{F}_p} = \boxed{\Omega_{\mathbb{F}_p[x]/\mathbb{F}_p} \otimes_{\mathbb{F}_p[x]} \mathbb{F}_p[x]/x^p.}$$

$$\Omega_{(\mathbb{Z}/p^n[x])/x^{p^n}} = \Omega_{\mathbb{Z}/p^n[x]} \otimes \mathbb{Z}/p^n[x]/x^{p^n}$$

$E/\mathcal{O}_{\mathbb{C}_p}$ an elliptic curve. (i.e. $E/\mathbb{C}_p$ w/ good reduction)

$\mathcal{O}$ ... $y^2 = x^3+ax+b$

$\mathcal{O}_{\mathbb{Q}_p}$ - valuation ring
$|z| \le 1$

$x^3 + \bar{a}x + \bar{b}$ no multiple roots in $\overline{\mathbb{F}_p}$.

Defined using polynomials (i.e. by a map of rings)

$$E[p^n] \times E[p^n] \to \mu_{p^n}$$

These are all schemes.

$$\mu_{p^n}(R) = \{ r \in R \text{ s.t. } r^{p^n} = 1 \}.$$
$$= \mathrm{Hom}(\mathbb{Z}[x]/x^{p^n}-1, R).$$

$$\mathbb{Z}[x]/x^{p^n}-1 =: \text{Functions on } \mu_{p^n}.$$

Only look at rings over $\mathcal{O}_{\mathbb{Q}_p}$.

If I take $t \in E[p^n](\mathcal{O}_{\mathbb{Q}_p})$.

$$\omega_t : (t, \ ) : E[p^n] \to \mu_{p^n}.$$
$$\omega_t^* : \Omega_{\mu_{p^n}/\mathcal{O}_{\mathbb{Q}_p}} \to \Omega_{E[p^n]/\mathcal{O}_{\mathbb{Q}_p}}.$$

because p is not invertible this has interesting information.

(E.g. $\mathbb{F}_3[x]/x^5-1$

$\Omega = 0.$ $\quad (x^3-1,\ 3x^2-1) = (1).$

$\parallel$

$\Omega_{\mathbb{F}_3[x]/(x^3-1)/\mathbb{F}_3}$

$\Omega_{\mu_{p^n}/\mathcal{O}_{\mathbb{C}_p}/p^n}.$

$\parallel$

$\mathcal{O}_{\mathbb{C}_p}/p^n[x]\Big/_{x^{p^n}-1} \cdot dx.$

$\wr$

specialize at $x=1$.

$\mathcal{O}_{\mathbb{C}_p}/p^n \cdot dx.$

Take limit as $n \to \infty$.

$\mathcal{O}_{\mathbb{C}_p} \cdot dx.$

$\parallel$

Cotangent space to $\mathbb{G}_m$ at $x=1$.

$\wr$

Same thing will happen for

$\Omega_{E[p^n]/\mathcal{O}_{\mathbb{C}_p}/p^n}.$

Specialize to the identity element in E.

$\parallel$

cotangent space to $\infty \in E$.

$$t \in T_p E = (t_{p^n})$$

$$\lim_{n \to \infty} (t_{p^n},\ )^* dx \Big|_\infty$$

$\|$

something in cotangents space of E at ∞. $\cong \mathcal{O}_{\mathbb{C}_p}$.

Defines a map

$$T_p E \to \mathcal{O}_{\mathbb{C}_p}.$$

Can show non-zero.

$$T_p E \otimes \mathbb{C}_p \twoheadrightarrow \mathbb{C}_p$$

2-dim'l 1-dim'l

HT filtration is the kernel.

Tate: p-divisible groups
(first paper on p-adic Hodge theory).