ON THE BOUNDEDNESS OF THE FUNCTOR OF KSBA STABLE VARIETIES

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Abstract. We show that all canonically polarised semi-log canonical pairs with fixed numerical invariants are of finitely many deformation types.

Contents

1. Introduction

Throughout this paper, we work over an algebraically closed field k of characteristic 0. Over the last few decades, the study of the moduli functor of families of KSBA-stable varieties, which provides a compacitification of the moduli functor of families of canonically polarized smooth varieties, has attracted a lot of

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interest. For simplicity, in the definition of the functor, we restrict ourselves to the case with no boundary.

Definition 1.1 (Moduli of slc models, cf. [Kollár13a, 29])**.** Let $H(m)$ be an integer valued function. The moduli functor of semi log canonical models with Hilbert function H is

$$\mathcal{M}^{slc}_H(S)=\left\{\begin{array}{c}\text{flat morphisms of proper families } X\to S,\\ \text{fibers are slc models with ample canonical class}\\ \text{and Hilbert function } H(m),\ \omega_X \text{ is flat over } S\\ \text{and commutes with base change.}\end{array}\right\}$$

We refer to the forthcoming book [Kollár14] for a detailed discussion of this subject and to [Kollár13a] for a more concise survey.

The aim of this note is to settle the question of the boundedness of $\mathcal{M}^{slc}_H$. In fact, we prove a more general statement, which we hope will also settle the boundedness for stable pairs, once the right functor has been defined.

Theorem 1.2. *Fix a positive integer $n\in\mathbb{N}$, a constant C and a DCC set $\mathcal{A}\subset[0,1]$. Consider the set*

$$F_{slc}(C,\mathcal{A})=\left\{(V,G)\left|\begin{array}{c}\textit{all } n\textit{-dimensional semi-log canonical projective}\\ \textit{pairs } (V,G=\sum g_iG_i),\textit{ such that } K_V+G \textit{ is ample,}\\ g_i\in\mathcal{A}\textit{ and } (K_V+G)^n=C.\end{array}\right.\right\}$$

Then $F_{slc}(C,\mathcal{A})$ has finitely many deformation types, i.e., there exists a variety S of finite type and a pair $(\mathcal{V},\mathcal{G})$ such that for any pair $(V,G)\in F_{slc}$, we can find a closed point $s\in S$ whose fiber (V_s,G_s) is isomorphic to (V,G).

Corollary 1.3. *Under the above notation. Assume we have a flat family of pairs*

$$(\mathcal{V},\mathcal{G})\to S$$

over a finite type scheme S such that for a dense set of points $p\in S$, the fibers $(V_p,G_p)\in F_{slc}(C,\mathcal{A})$. Then there is an dense open set U, such that for any closed point $t\in U$, the fiber $(V_t,G_t)\in F_{slc}(C,\mathcal{A})$.

We remark that the strategy of proving Theorem 1.2 goes back to Kollár [Kollár94], and when $n=2$ it was first worked out by Alexeev (see [Alexeev94, AM04]).

Recall that (cf. [Kollár11] or [Kollár13b, 5.13]), the datum of an slc canonically polarized pair (V,G) is equivalent to that of a quadruple

$$(\bar{V},\bar{G},\bar{D},\tau)$$

such that

(1) $\bar{V}=\coprod_s V^s$ is the normalization of V; $\bar{G}$ (resp. $\bar{D}$) is the preimage of G (resp. of the self-intersection locus of V);

(2) $\tau : \bar{D}^n \to \bar{D}^n$ is an involution on the normalization $\bar{D}^n$ of $\bar{D}$ and $\mathrm{Diff}_{\bar{D}^n}\bar{G}$ is τ-invariant, and
(3) $(\bar{V}, \bar{G} + \bar{D})$ is a canonically polarized (possibly non-connected) lc pair.

A large part of this theorem, namely, showing that the normalization $(\bar{V}, \bar{G}, \bar{D})$ of (V, G) has finitely many possible numerical invariants, has been worked out in the authors' previous work [HMX12]. Furthermore, the techniques developed in [HMX12] also show that the set of all triples $(\bar{V}, \bar{G}, \bar{D})$ as above, is birationally bounded. Therefore, to prove boundedness, the key issue is to prove the following "generic abundance" theorem (1.4), which is the main technical result of this paper.

Theorem 1.4. *Let (X, Δ) be a dlt pair and $\pi : X \to S$ a projective morphism to an integral variety S. Assume that there is a dense set of closed points $\{t_j\}$ of S such that the fibers (X_j, Δ_j) have a good model. Then there is a dominant generic finite morphism $T \to S$, such that $(X, \Delta) \times_S T$ has a good model over T.*

Finally, to show (1.2) for slc pairs, we invoke Kollár's theorem (cf. [Kollár11]) stating that slc canonically polarized pairs (V, Δ) are given by possibly non-connected log canonical polarized pairs $(\bar{V}, \bar{\Delta} + \bar{D})$ (corresponding to the normalization of (V, Δ)) together with certain glueing data $\tau : \bar{D}^n \to \bar{D}^n$. The case of slc pairs then follows from the case of lc pairs once we show that the glueing data is bounded.

2. Preliminaries

Notations and Conventions: We will follow the terminology from [KM98]. We will also need the definition of certain singularities of semi-normal pairs. Let X be a semi-normal variety which satisfies Serre's condition S_2 and Δ be a $\mathbb{Q}$-divisor on X, such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $n : X^n \to X$ be the normalization of X and write $n^*(K_X + \Delta) = K_{X^n} + \Delta^n + \Gamma$, where Γ is the reduced double locus. We say that (X, Δ) is *semi-log canonical* or *slc* if $(X^n, \Delta^n + \Gamma)$ is log canonical and (X, Δ) is *divisorial semi-log-terminal* or *dslt* if $(X^n, \Delta^n + \Gamma)$ is dlt. Note that if (X, Δ) is dlt and B is a union of components of $\lfloor \Delta \rfloor$, then $(B, \mathrm{Diff}^*_B(\Delta))$ is *dslt*, where $K_B + \mathrm{Diff}^*_B(\Delta) = (K_X + \Delta)|_B$.

Let $\phi : X \dashrightarrow Y$ be a proper birational contraction of normal quasi-projective varieties (so that in particular ϕ^{-1} contracts no divisors). If D is a $\mathbb{Q}$-Cartier divisor on X such that $D' := \phi_* D$ is $\mathbb{Q}$-Cartier then we say that ϕ is *D-non-positive* (resp. *D-negative*) if for a common resolution $p : W \to X$ and $q : W \to Y$, we have $p^*D = q^*D' + E$ where $E \geq 0$ and p_*E is ϕ-exceptional (respectively p_*E and its support equals the set of ϕ-exceptional divisors). If $f : X \to U$ and $f_M : X_M \to U$ are projective morphism, $\phi : X \dashrightarrow X_M$ is a birational contraction and (X, Δ) and (X_M, Δ_M) are log canonical pairs (resp. klt pairs, dlt pairs) such that $a(E; X, \Delta) > a(E; X_M, \Delta_M)$ (resp. $a(E; X, \Delta) \geq a(E; X_M, \Delta_M)$) for all ϕ-exceptional divisors $E \subset X$, X_M is $\mathbb{Q}$-factorial and $K_{X_M} + \Delta_M$ is nef over U, then we say that $\phi : X \dashrightarrow X_M$ is *a minimal model* of $K_X + \Delta$ over U.

We say $K_{X_M}+\Delta_M$ is *semi-ample* over U if there exists a surjective morphism $\psi : X_M \to Z$ over U such that $K_{X_M}+\Delta_M \sim_{\mathbb{Q}} \psi^*A$ for some $\mathbb{Q}$-divisor A on Y which is ample over U. Equivalently $K_{X_M}+\Delta_M$ is semi-ample over U if there exists an integer $m>0$ such that $\mathcal{O}_{X_M}(m(K_{X_M}+\Delta_M))$ is generated over U. Note that in this case

$$R(X_M/U, K_{X_M}+\Delta_M) := \bigoplus_{m\geq 0} f_*\mathcal{O}_{X_M}(m(K_{X_M}+\Delta_M))$$

is a finitely generated $\mathcal{O}_U$-algebra, and $Z = \mathrm{Proj}R(X_M/U, K_{X_M}+\Delta_M)$. Recall that for any $\mathbb{Q}$-divisor D on X, the sheaf $f_*\mathcal{O}_X(D)$ is defined to be $f_*\mathcal{O}_X(\lfloor D\rfloor)$. A minimal model $\phi : X \dashrightarrow X_M$ is called *a good minimal model* (or *a good model* for short) if $K_{X_M}+\Delta_M$ is semi-ample. If $K_{X_M}+\Delta_M$ is semi-ample and big over U, then we let $X_{LC} = \mathrm{Proj}R(X/U, K_X+\Delta)$ be the *log canonical model* of (X,Δ) over U. More generally, we say that a birational contraction $g : X \dashrightarrow Y$ over U is a *semi-ample model* of a $\mathbb{Q}$-Cartier divisor D over U if g is D-non-positive, Y is normal and projective over U and $H = g_*D$ is semi-ample over U.

Let D be a $\mathbb{Q}$-Cartier divisor on a projective X, following [Nakayama05] we define *the numerical dimension*

$$\kappa_\sigma(D) := \max_{H\in \mathrm{Pic}(X)}\{k\in\mathbb{N} | \limsup_{m\to\infty} \frac{h^0(X, mD+H)}{m^k} > 0\}.$$

When D is nef, it is the same as (see [Nakayama05])

$$\nu(D) = \max\{k | H^{n-k}\cdot D^k > 0\}$$

for any ample divisor H. In general, if $f : X \to U$ is a projective morphism, and D is $\mathbb{Q}$-divisor on X. We say that D is *f-abundant* if restricting on the generic fiber we have $\kappa_\sigma(D|_{X_\eta}) = \kappa(D|_{X_\eta})$, i.e. the numerical dimension is equal to the Iitaka dimension.

Let $g : X \to Y$ be a proper morphism between two normal varieties. We say that g is an *algebraic fibration* if $g_*\mathcal{O}_X = \mathcal{O}_Y$.

Let (X,Δ) a pair, then the non-canonical locus of $\mathrm{Ncan}(X,\Delta)$ is the union of the centers on X of divisors E over X with discrepancy $a(E;X,\Delta)<0$.

2.1. **Invariance of plurigenera.**

We will need the following result of B. Berndtsson and M. Paun.

Theorem 2.1. *Let $f : X \to S$ be a projective morphism from a smooth variety to the unit disk S. Assume that*

1. *Δ is a $\mathbb{Q}$-divisor on X with $\lfloor\Delta\rfloor$ and whose components Δ^i are smooth and disjoint,*
2. *$0\in S$ is a closed point such that $X_0+\sum\Delta^i$ has simple normal crossings,*
3. *$K_X+\Delta$ is pseudo-effective and $\mathbf{B}_-(K_X+\Delta)$ contains no component of $\Delta^i_0 := \Delta^i|_{X_0}$.*

Then $H^0(X, \mathcal{O}_X(m(K_X+\Delta))) \to H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0}+\Delta_0)))$ *is surjective for any integer* m *such that* $m\Delta$ *is integral.*

Proof. This is an immediate consequence of [BP10, Thm. 0.2]. We will now check that the hypothesis of [BP10, Thm. 0.2] are satisfied. We choose $L = \mathcal{O}_X(m\Delta)$ so that $m[\Delta] \in c_1(L)$. Note that by assumption (3), $K_X + \Delta$ is pseudo-effective and $\nu_{\min}(\{K_X+\Delta\}, X_0) = 0$ and $\rho^j_{\min,\infty} = 0$. In particular $J = J'$. Let $u \in H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0}+\Delta_0)))$ be any non-zero section, then we choose $h_0 = e^{-\varphi_0}$ such that $\varphi_0 \le 0$ and $\Theta_{h_0}(K_{X_0}+\Delta_0) = \frac{1}{m}[Z_u]$. Since u has no poles and $\lfloor \Delta \rfloor = 0$, we have $\int_{X_0} e^{\varphi_0 - \frac{1}{m}\varphi_{m\Delta}} < \infty$. Condition $(\star)$ of [BP10, Thm. 2] is also satisfied as $J = J'$ and $\rho_{\min,\infty} = 0$. Thus there exists $U \in H^0(X, \mathcal{O}_X(m(K_X+\Delta)))$ which extends u. □

Theorem 2.2. *Let* (X, Δ) *be a klt pair, and* $f : X \to C$ *a projective morphism to a smooth curve* C*, such that* $(X, \operatorname{Supp}(\Delta))$ *is log smooth over* C*. If* $0 \in C$ *is a closed point, then*

$$f_*\mathcal{O}_X(m(K_X+\Delta)) \to H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0}+\Delta_0)))$$

is surjective for all sufficiently divisible integers $m > 0$*.*

Proof. If Δ is big, the theorem follows from [HMX13, 1.6]. Replacing X by an appropriate birational model, we may assume that the components of Δ are disjoint. Let $\Delta'_0 = \Delta_0 - (\Delta_0 \wedge \mathbf{B}_-(K_{X_0}+\Delta_0))$. It follows easily from [BCHM10] that $N_\sigma(K_{X_0}+\Delta_0)$ is a $\mathbb{Q}$-divisor and hence that Δ'_0 is a $\mathbb{Q}$-divisor. Let $0 \le \Delta' \le \Delta$ be the corresponding $\mathbb{Q}$-divisor, so that $\Delta'|_{X_0} = \Delta'_0$. It suffices to show that

$$f_*\mathcal{O}_X(m(K_X+\Delta')) \to H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0}+\Delta'_0)))$$

is surjective for all sufficiently divisible integers $m > 0$. Replacing Δ by Δ' we may thus assume that $\Delta_0 \wedge \mathbf{B}_-(K_{X_0}+\Delta_0) = 0$. By (2.1), it suffices to show that $\mathbf{B}_-(K_X+\Delta/S)$ contains no components of Δ_0.

We run a $(K_X+\Delta)$-MMP over S with scaling of an ample divisor H and we obtain a sequence of flips and divisorial contractions $g^k : X^k \dashrightarrow X^{k+1}$ and rational numbers $1 = s_0 \ge s_1 \ge s_2 \ge \dots$ such that $K_{X^i} + \Delta^i + sH^i$ is nef over S for any $s_i \ge s \ge s_{i+1}$ and either the sequence is infinite in which case we have $\lim s_i = 0$ or finite in which case we have $i \in \{0, 1, \dots, N\}$ and either $s_N = 0$ or $K_X + \Delta$ is not pseudo-effective over S.

By the proof of [HMX13, 1.8] (also see the proof of (3.1)), we may assume that

(1) if g^k contracts a component B of Δ^k, then g^k_0 contracts B_0,
(2) the indeterminacy locus of g^k does not contain any components of Δ^k_0, and
(3) g^k_0 is a birational contraction.

In particular this yields a sequence of weak log canonical models for $K_{X_0} + \Delta_0 + s_iH_0$. Since $K_{X_0} + \Delta_0 + sH_0$ is big for any $s > 0$ it follows that if the sequence is finite, then $s_N = 0$.

Since $K_{X^i} + \Delta^i + sH^i$ is semiample for any $s > 0$ such that $s_i \geq s \geq s_{i+1}$, it follows easily that $\mathbf{B}_{-}(K_X + \Delta/S)$ is the union of the indeterminacy loci of $\phi^i : X \dashrightarrow X^i$. Thus, by (2) above, no component of Δ^i_0 is contained in the indeterminacy locus of g^i.

□

We have the following corollary, which is also proved in [HMX13, 1.8].

Corollary 2.3. *Let $f : (X, \Delta) \to S$ be a lc pair, which is projective over S, such that $(X,\ \mathrm{Supp}(\Delta))$ is log smooth over S. Then the numerical dimension $\kappa_\sigma(K_{X_s} + \Delta_s)$ is a constant for any $s \in S$.*

3. Minimal models

We start to prove (1.4). In this section we first verify the existence of a minimal model for the generic fiber, under the assumption that $(1.4)_{n-1}$ is true.

3.1. Base Change.

Lemma 3.1. *Let (X, Δ) be a dlt pair and $f : X \to C$ a projective morphism to a smooth curve, which satisfies*

(1) *(X_s, Δ_s) is a dlt pair for any $s \in C$,*
(2) *for each component Δ_i of Δ, $\Delta_i \cap X_s$ is irreducible for any $s \in S$, and*
(3) *$\mathrm{Ncan}(X_s, \Delta_s) \cap \mathbf{B}_{-}(K_X + \Delta/S) = \emptyset$ for any s.*

Let H be an ample divisor, and $m > 0$ an integer such that $m(K_X + \Delta)$ is an integral Weil divisor, then

$$f_*\mathcal{O}_X(m(K_X + \Delta) + H) \to H^0(X_s, \mathcal{O}_{X_s}(m(K_{X_s} + \Delta_s) + H_s))$$

is surjective.

Proof. This result is essentially proven in Section 4 of [HMX13]. We include a proof for the reader's convenience.

We run the $(K_X + \Delta)$-MMP over C with scaling of H. Let $g^i : X^i \dashrightarrow X^{i+1}$ be the corresponding sequence of flips and divisorial contractions. Let $s \in C$ be any closed point.

Claim 3.2. For all $i \geq 0$ we have

(1) g^i contracts no component B of Δ^i,
(2) the indeterminacy locus of g^i contains no components of Δ^i_s, and
(3) g^i_s is a birational contraction.

Proof. We proceed by induction on i and hence we may assume that $(1\text{-}3)_j$ hold for all $0 \leq j < i$. In particular if B is an irreducible component of Δ, then B^i_s is irreducible.

Suppose that g^i contracts a component B of Δ^i, then $V := g^i(B)$ is irreducible, dominates C and the dimension of the fibers of $B \to V$ is upper-semicontinuous.

Thus g^i_s contracts B_s. But then B_s is contained in $\mathrm{Ncan}(X_s,\Delta_s)\cap\mathbf{B}_-(K_X+\Delta/S)$ contradicting our assumptions. Thus $(1)_i$ holds

Suppose that an irreducible component B_s of Δ^i_s is contained in the indeterminacy locus, then g^i is a flip and the corresponding flipping contraction $\pi : X^i \to Z$ contracts B_s. By semicontinuity of the dimension (applied to $\pi(B)\to C$), it follows that π contracts B, where B is the corresponding irreducible component of Δ^i. This is impossible and hence $(2)_i$ holds.

If g^i_s is not a birational contraction, then there is a divisor P on X^{i+1}_s whose center V on X^i_s has codimension ≥ 2. By our assumptions we have that $V\cap \mathrm{Ncan}(X^i_s,\Delta^i_s)=\emptyset$ and so $a(P;X^i_s,\Delta^i_s)\geq 0$. We also have

$$0\geq a(P;X^{i+1}_s,\Delta^{i+1}_s)>a(P;X^i_s,X^i_s),$$

which is impossible and so $(3)_i$ holds. □

By [BCHM10], there exists $n\geq 0$ such that $K_{X^n}+\Delta^n+\frac{1}{m}H^n$ is nef over C. Note also that there exists a $\mathbb{Q}$-divisor $\Theta_m\sim_{\mathbb{Q},C}\Delta+\frac{1}{m}H$ such that (X,Θ_m) is klt and hence so is (X^n,Θ^n_m). By the Kawamata-Viehweg vanishing theorem (cf. [KM98, 2.70]), it follows that

$$f_*\mathcal{O}_{X^n}(m(K_{X^n}+\Delta^n)+H^n)\to H^0(X^n_s,\mathcal{O}_{X^n_s}(m(K_{X^n_s}+\Delta^n_s)+H^n_s))$$

is surjective for any $m>0$, such that $m(K_{X^n}+\Delta^n)+H^n$ is an integral Weil divisor. We have the following commutative diagram

$$\begin{tikzcd}
f_*\mathcal{O}_X(m(K_X+\Delta+H)) \arrow[r] \arrow[d, "\cong"'] & H^0(X_s,\mathcal{O}_{X_s}(m(K_{X_s}+\Delta_s+H_s))) \arrow[d, hook] \\
f_*\mathcal{O}_{X^n}(m(K_{X^n}+\Delta^n+H^n)) \arrow[r, two heads] & H^0(X^n_s,\mathcal{O}_{X^n_s}(m(K_{X^n_s}+\Delta^n_s+H^n_s))),
\end{tikzcd}$$

where the right vertical arrow is an injection by (3) of the above claim. Therefore, we conclude, that

$$f_*\mathcal{O}_X(m(K_X+\Delta+H))\to H^0(X_s,\mathcal{O}_{X_s}(m(K_{X_s}+\Delta_s+H_s)))$$

is surjective. □

To settle the case of higher dimensional base, we need the following lemma.

Lemma 3.3. *Let $f:X\to S$ be a proper flat morphism to a smooth variety S, and L a line bundle on X. Fix a point $s\in S$, and assume that for any general curve C containing s, $f_{C*}(L|_{X_C})\otimes k(s)\to H^0(X_s,L|_{X_s})$ is an isomorphism. Then the base change morphism*

$$f_*(L)\otimes_{\mathcal{O}_S}\mathcal{O}_C\to f_{C*}(L|_{X_C})$$

is an isomorphism, where $X_C\cong X\times_S C$.

Proof. It suffices to show that $f_*(L)\otimes_{\mathcal{O}_S}\mathcal{O}_C \to f_{C*}(L|_{X_C})$ is an isomorphism on a neighborhood of s. We may assume that $S = \mathrm{Spec}(A)$ is an affine scheme where (A, m) is a local ring and $k(s) = A/m$. By [Hartshorne77, III, 12.4], there exists a finitely generated A-module Q such that for any A-module M, $H^0(X, L \otimes_A M) \cong \mathrm{Hom}_A(Q, M)$. Let I be the ideal of $C \subset S$, then we must show that the homomorphism $\mathrm{Hom}_A(Q, A) \otimes_A A/I \to \mathrm{Hom}_A(Q, A/I)$, is an isomorphism. Clearly it suffices to show that Q is locally free or equivalently that Q is a flat A-module.

We begin by showing that Q is torsion free. If this were not the case, then $Q \otimes_A A/I$ is also not torsion free for general C and so

$$\mathrm{Hom}_{A/I}(Q \otimes_A A/I, A/I) \otimes_A A/m \to \mathrm{Hom}_{A/m}(Q \otimes A/m, A/m)$$

is not surjective. However, the hypothesis that $f_{C*}(L|_{X_C})\otimes_{\mathcal{O}_S} k(s) \to H^0(X_s, L|_{X_s})$ is an isomorphism is equivalent to the homomorphism $\mathrm{Hom}_A(Q, A/I)\otimes_A A/m \to \mathrm{Hom}_A(Q, A/m)$ being an isomorphism. This is a contradiction and so Q is torsion free.

We now conclude the proof by induction on $\dim S$. Let H be a general very ample divisor through s. By induction, we may assume that $Q|_H$ is flat on H. By the "local criterion of flatness" (cf. [Hartshorne77, III.10.3.A]), we conclude that Q is a flat A module. $\square$

Proposition 3.4. *Let (X, Δ) be a dlt pair and $f : X \to S$ an equi-dimensional projective morphism over a smooth variety, which satisfies the analogs of the conditions in (3.1). Let $g^i : X^i \dashrightarrow X^{i+1}$ be a $(K_X + \Delta)$-divisorial contraction or flip over S and $s \in S$ a closed point. Then*

1. *g^i contracts no component B of Δ^i,*
2. *the indeterminacy locus of g^i contains no components of Δ^i_s, and*
3. *$X^i_s \dashrightarrow X^{i+1}_s$ is a birational contraction. In particular, $X' \to S$ is equi-dimensional.*

Proof. The proof of (1) and (2) are the same for higher dimensional base as the case for the base being a curve (cf. the proof of (3.1)). Thus, we only need to verify (3).

Note that since X^i is Cohen-Macaulay, then X is flat over S (cf. [Hartshorne77, III. Ex 10.9]). We can assume that $S = \mathrm{Spec}(A)$ is affine. Let $X^i \to Z$ be the flipping contraction, so that $Z = \mathrm{Proj}R(X/S, K_X + \Delta + tH)$ for some ample divisor H and some rational number $t > 0$. If $X^i \dashrightarrow X^{i+1}$ is a flip, then for some $0 < \epsilon \ll 1$, we have that

$$X^{i+1} = \mathrm{Proj}R(X/S, K_X + \Delta + (t - \epsilon)H).$$

We can assume d is a sufficiently big positive integer such that the truncations $R^d(X/S, K_X + \Delta + tH)$, $R^d(X_{k(s)}/k(s), K_{X_{k(s)}} + \Delta_{k(s)} + tH_{k(s)})$, $R^d(X/S, K_X + \Delta + (t - \epsilon)H)$ and $R^d(X_{k(s)}/k(s), K_{X_{k(s)}} + \Delta_{k(s)} + (t - \epsilon)H_{k(s)})$ are generated by elements of degree d. We note that the finite generation of these rings follows from [BCHM10].

Let $C \to S$ be a general smooth curve passing through s.

Claim 3.5. Denote by $\bullet_C$ the base change of $\bullet$ over C, then

$$Z_C \cong \mathrm{Proj} R(X_C/C, K_{X_C} + \Delta_C + tH_C).$$

In fact, by definition Z_C is $\mathrm{Proj} R(X/S, K_X + \Delta + tH) \otimes_{\mathcal{O}_S} \mathcal{O}_C$. Thus, it suffices to check that the assumptions of (3.3) hold for X/S, C and $L = d(K_X + \Delta + tH)$. However, this follows from (3.1).

Similarly, in the flip case, we also have that

$$X_C^{i+1} \cong \mathrm{Proj} R(X_C/C, K_{X_C} + \Delta_C + (t-\epsilon)H_C).$$

But then it is easy to see that $X_C^{i+1} \to Z_C$ is indeed a small morphism (cf. [KM98, 6.2]). Thus (3) holds. □

3.2. Minimal models over the generic point.

Theorem 3.6. *Let (X, Δ) be a dlt pair, $\pi : X \to S$ a projective morphism, $\eta \in S$ the generic point and $\{t_j\}_{j \in \mathbb{N}} \subset S$ a dense set of points such that the fibers $(X_j, \Delta_j) := (X, \Delta) \times_S \{t_j\}$ are dlt pairs and have a good minimal model. Replacing S by a dominant generically finite base change, there exists a $(K_X + \Delta)$-MMP (with scaling over S), $X = X^0 \dashrightarrow X^1 \dashrightarrow \cdots \dashrightarrow X^n$ such that $(X_\eta^n, \Delta_\eta^n)$ is a $(K_{X_\eta} + \Delta_\eta)$ minimal model, where η is the generic point of S and $\bullet_\eta$ denotes base change over η.*

Proof. We start with the following lemma.

Lemma 3.7. *we may assume that*

(1) $\lfloor \Delta \rfloor \cap \mathbf{B}_-(K_X + \Delta/S) = \emptyset$ *and that*
(2) $(K_X + \Delta)|_{X \setminus \lfloor \Delta \rfloor}$ *is canonical.*

Proof. We run a $(K_X + \Delta)$-MMP with over S with scaling of a general sufficiently ample divisor A. Let $g^i : X^i \dashrightarrow X^{i+1}$ be the corresponding flips and contractions so that $K_{X^i} + \Delta^i + tA^i$ is nef for $s_i \geq t \geq s_{i+1}$. We may assume that g^i is a flip for any $i \gg 0$, and by a discrepancy computation, we may assume that the induced rational maps $\lfloor \Delta^i \rfloor \dashrightarrow \lfloor \Delta^{i+1} \rfloor$ are isomorphisms in codimension 1. In particular, we may also assume (after possibly rechoosing A) that the restriction of A^i to any irreducible component G^i of $\lfloor \Delta^i \rfloor$ is big over S. Then $K_{G^i} + \Delta_{G^i} := (K_{X^i} + \Delta^i)|_{G^i}$ is dlt and $K_{G^i} + \Delta_{G^i} + tA_{G^i}$ is nef over S where $A_{G^i} = A^i|_{G^i}$. For any rational number $0 < \epsilon \leq s_i$, there exists a $\mathbb{Q}$-divisor

$$\Theta = \Theta(G, \epsilon) \sim_{\mathbb{Q}, S} \Delta_G + \epsilon A_G$$

whose support contains the support of A_G and such that $K_G + \Theta$ is klt. Since $G \dashrightarrow G^i$ is a $(K_G + \Theta)$-non-positive birational contraction, then $K_{G^i} + \Theta_{G^i}$ is also klt and

$$K_{G^i} + \Theta_{G^i} \sim_{\mathbb{Q}, S} K_{G^i} + \Delta_{G^i} + \epsilon A_{G^i}.$$

After shrinking S, we may assume that each component G^i dominates S.

By induction on the dimension of X and (1.4), (after shrinking S) we may assume that $K_{G^i}+\Delta_{G^i}$ has a good minimal model over S say $\psi^i : G^i \dashrightarrow \bar{G}$. Let $\bar{G} \to \bar{D} = \mathrm{Proj} R(K_{G^i}+\Delta_{G^i}/S)$ be the corresponding morphism. Let $\Delta_{\bar{G}}$ (resp. $\bar{A}$) be the pushforward of Δ_{G^i} (resp. A_{G^i}) to $\bar{G}$. By [HX13, 2.9], we may assume that ψ^i is given by a $\mathbb{Q}$-factoralization of $\bar{G}^i \to G^i$ followed by a sequence of $K_{\bar{G}^i}+\Delta_{\bar{G}^i}$ flips and divisorial contractions so that ψ^i is an isomorphism at the generic point of each strata of each non-klt center of $(\bar{G}, \Delta_{\bar{G}})$ (cf. [BCHM10, 3.10.11]). Thus, $(\bar{G}, \Delta_{\bar{G}}+s_i\bar{A})$ is dlt with the same non-klt centers as $(\bar{G}, \Delta_{\bar{G}})$.

Claim 3.8. For $0<\epsilon \ll 1$ there is a good minimal model over $\bar{D}$ for $(\bar{G}, \Delta_{\bar{G}}+\epsilon\bar{A})$.

Proof. Since $K_{G^i}+\Delta_{G^i}+\epsilon A_{G^i} \sim_{\mathbb{Q},S} K_{G^i}+\Theta_{G^i}$ where (G^i, Θ_{G^i}) is klt and Θ_{G^i} is big (as its support contains the support of A_{G^i}), the result follows from [BCHM10, 1.4.2]. $\square$

By the arguments of [HX13, 5.8], we may assume that each flip g^i is disjoint from $\lfloor \Delta^i \rfloor$. Since for any $1 \gg \epsilon > 0$ there is an $i>0$ such that $K_{X^i}+\Delta^i+\epsilon A^i$ is semiample over U, replacing X by X^i, (1) follows easily.

Then we can obtain (2) by applying [BCHM10, 1.4.3] to extract finitely divisors, whose centers are contained in $X\setminus \lfloor \Delta \rfloor$ and have negative discrepancies. $\square$

After shrinking the base S, we can assume S is smooth, (X_s, Δ_s) is a dlt pair for any $s\in S$. After a base change $T\to S$ (and without changing the notation), we may further assume that for any $s\in S$ and any component B of $\mathrm{Supp}(\Delta)$, the fiber B_s is irreducible. Therefore, $f:(X,\Delta)\to S$ satisfies the conditions of (3.4) and so the $(K_X+\Delta)$-MMP with scaling over S yields a sequence of weak log canonical models for $K_{X_j}+\Delta_j+s_iA_j$. Choose the point $s=t_1$. Since (X_s,Δ_s) has a good minimal model, all flips and contractions are eventually disjoint from X_s and hence $K_{X^i_\eta}+\Delta^i_\eta$ is nef for $i >> 0$. In other words, if we let η be the generic point of S, then $(K_{X_\eta}+\Delta_\eta)$ has a minimal model. $\square$

4. Generic Abundance

In the section, we finish the proof of (1.4).

After replacing S by a dominating base change, we may assume that for any $s\in S$ and any component B of $\mathrm{Supp}(\Delta)$, the fiber B_s is irreducible and by (3.6) we can assume that $K_{X_\eta}+\Delta_\eta$ is nef. We aim to verify that $K_{X_\eta}+\Delta_\eta$ is semiample. Replacing S by a non-empty open subset, we can assume that (X,Δ) has a log resolution, which is log smooth over S. Therefore, it follows from (2.2) and (2.3) that we can assume:

(1) The numerical dimension $\kappa_\sigma(K_{X_s}+\Delta_s)$ is constant for any $s\in S$.
(2) Let (X,Δ') be a klt pair, such that the support of Δ' is contained in the support of Δ, then $h^0(X_s, \mathcal{O}_{X_s}(m(K_{X_s}+\Delta'_s)))$ is a constant function of $s\in S$ for any sufficiently divisible integer $m>0$.

4.1. **Numerical dimension equal to Kodaira dimension.** In this step, we will show that:

Proposition 4.1. *With the above notation, we have $\kappa(K_{X_\eta}+\Delta_\eta)=\nu(K_{X_\eta}+\Delta_\eta)$.*

Proof. We may assume that $\kappa(K_{X_\eta}+\Delta_\eta)\le\nu(K_{X_\eta}+\Delta_\eta)<\dim X_\eta$. Let $f_j: X_j \dashrightarrow X^m_j$ be a good minimal model of (X_j,Δ_j) and

$$h_j: X^m_j \to Z_j := \mathrm{Proj} R(X_j, K_{X_j}+\Delta_j)$$

be the induced morphism. Write $\Delta_j = F(j)+\Gamma(j)$, where $\Gamma(j)$ precisely consists of those components which are not contracted by f_j and dominate Z_j. Since Δ has finitely many components, passing to a (dense) subsequence, we may assume that $\Delta = F+\Gamma$, where $\Gamma|_{X_j}=\Gamma(j)$ and $F|_{X_j}=F(j)$ for all j. We first assume that $\Gamma\neq 0$.

Lemma 4.2. *For any $1>\epsilon>0$, denote by $\Delta_\epsilon := (1-\epsilon)\Gamma+F$. Assume $\Gamma\neq 0$, then $K_{X_\eta}+\Delta_\epsilon|_{X_\eta}$ is not pseudo-effective.*

Proof. Consider a log resolution of $\pi: X'\to X$ of (X,Δ) and write

$$p^*(K_X+(1-\epsilon)\Gamma+F)+E = K_{X'}+\Delta'_\epsilon,$$

where E, Δ'_ϵ are effective and have no common components. By shrinking S, we may assume (X',Δ'_ϵ) is log smooth over S. Since $K_{X'_j}+\Delta'_\epsilon|_{X_j}$ is not pseudo-effective, then it follows from (2.3) that $K_{X'_\eta}+\Delta'_\epsilon|_{X'_\eta}$ is not pseudo-effective and the lemma follows immediately. □

Lemma 4.3. *Let N be a positive integer, such that $N(K_{X_\eta}+\Delta_\eta)$ is Cartier. Let $\epsilon<\frac{1}{1+2N\dim X}$, then*

1. *restricting to the generic fiber, any step $X^i \dashrightarrow X^{i+1}$ of the $(K_X+\Delta_\epsilon)$-MMP is $(K_{X^i_\eta}+\Delta^i_\eta)$-trivial, and*
2. *$K_{X^i_\eta}+\Delta^i_\eta$ is nef and $N(K_{X^i_\eta}+\Delta^i_\eta)$ is Cartier.*

Proof. We proceed by induction on i. We will show that $(1)_{i-1}$ and $(2)_i$ imply $(1)_i$ and $(2)_{i+1}$. If the divisorial/filpping contraction does not intersect with X_η (1) and (2) are obvious. Otherwise, the contraction is defined by an extremal ray spanned by a curve $[C]\in NE_1(X^i_\eta)$. Since $K_{X^i_\eta}+\Delta^i_\eta$ is nef, we have $(K_{X^i_\eta}+\Delta^i_\eta)\cdot C\ge 0$. If $(K_{X^i_\eta}+\Delta^i_\eta)\cdot C>0$, then as $N(K_{X^i_\eta}+\Delta^i_\eta)$ is Cartier, it follows that $(K_{X^i_\eta}+\Delta^i_\eta)\cdot C\ge\frac{1}{N}$. Since C is a $(K_{X^i_\eta}+F^i_\eta)$-negative extremal ray, we may assume that $-(K_{X^i_\eta}+F^i_\eta)\cdot C\le 2\dim(X)$ (cf. [Kawamata91]). This yields a contradiction, since $\epsilon<\frac{1}{1+2N\dim X}$, and so $(1)_i$ holds. Finally $(2)_{i+1}$ follows $(1)_i$ and $(2)_i$ as an immediate consequence of [KM98, 3.17]. □

Therefore, if we run the $(K_X+\Delta_\epsilon)$-MMP over S with scaling of an ample divisor, we obtain a sequence of divisorial contractions or flips which are all $(K_{X^i_\eta}+\Delta^i_\eta)$-trivial,

$$X = X^0 \dashrightarrow X^1 \dashrightarrow \cdots \dashrightarrow X^n,$$

and $g: X^n \to Y$ a Fano contraction such that $K_{X^n_\eta} + \Delta^n_\eta \sim_{\mathbb{Q}} g^*L$ for some $\mathbb{Q}$-Cartier divisor L on Y. By shrinking S, we can assume $X \dashrightarrow X^n$ is $(K_X+\Delta)$-trivial over S. Therefore, we replace X by X^n (without changing the notation).

Lemma 4.4. *If Γ has a component with coefficient 1, then $K_{X_\eta} + \Delta_\eta$ is semi-ample.*

Proof. Let B be such a component. Write $(K_X + \Delta)|_B = K_B + \Delta_B$. Since $(K_B + \Delta_B)|_{B_\eta}$ is nef, $\dim(B) < \dim(X)$ and by (3.7) over s_j the fiber (B_j, Δ_{B_j}) of (B, Δ_B) has a good minimal model, it follows by induction on the dimension that $(K_B + \Delta_B)|_{B_\eta}$ is semi-ample. Since

$$(K_B + \Delta_B)|_{B_\eta} \sim_{\mathbb{Q}} (g|_{B_\eta})^* L,$$

this implies that L is semi-ample. □

Therefore, to conclude the proof, we may assume that all coefficients of Γ are less than 1 so that $\Gamma = 0$. Let $t_j \in S$ be a point such that Z_j is not dominated by any component of F_j. Therefore, for a fixed j, there exists a sufficiently small $\epsilon > 0$, such that $(X_j, \Gamma_j + (1-\epsilon)F_j)$ has the same Kodaira dimension as (X_j, Δ_j). It follows that

$$\begin{aligned}\kappa(K_{X_\eta} + \Gamma_\eta + F_\eta) &\geq \kappa(K_{X_\eta} + \Gamma_\eta + (1-\epsilon)F_\eta) = \kappa(K_{X_j} + \Gamma_j + (1-\epsilon)F_j)\\ &= \kappa(K_{X_j} + \Gamma_j + F_j) = \kappa_\sigma(K_{X_j} + \Gamma_j + F_j) = \nu(K_{X_\eta} + \Gamma_\eta + F_\eta).\end{aligned}$$

The first inequality holds as F_η is effective, the second equality holds by equation (2) at the beginning of §4 (note that $(X_j, \Gamma_j + (1-\epsilon)F_j)$ is klt as $(X_j, \Gamma_j + F_j)$ is dlt and $\lfloor \Gamma_j + (1-\epsilon)F_j \rfloor = 0$), the third equality holds as F_j is either f_j-exceptional or h_j-vertical. This concludes the proof. □

4.2. **Abundance for semi-log canonical varieties.** Let (X, Δ) be a projective dlt pair and write $\Delta = \Gamma + \Theta$ where $\Gamma = \lfloor \Delta \rfloor$ is the non-klt locus. Thus $(\Gamma, \mathrm{Diff}_\Gamma \Theta)$ is a dslt pair. Let $\pi : \Gamma^n = \coprod \Gamma^i \to \Gamma$ be the normalization and write

$$\pi^*(K_\Gamma + \mathrm{Diff}_\Gamma \Theta) = K_{\Gamma^n} + \mathrm{Diff}_{\Gamma^n} \Theta + E,$$

where E is the double locus. Then $(\Gamma^n, \mathrm{Diff}_{\Gamma^n}\Theta + E)$ is a projective (not necessarily connected) dlt pair. We have the following result.

Theorem 4.5. *If $K_{\Gamma^n} + \mathrm{Diff}_{\Gamma^n}\Theta + E$ is semi-ample, then $K_\Gamma + \mathrm{Diff}_\Gamma \Theta$ is semi-ample.*

Proof. See [FG11] or [HX11, 1.4]. □

4.3. **Base Point Free Theorem.** Recall the following generalization of Kawamata's theorem.

Theorem 4.6. *Let (X, Δ) be a dlt pair, H a Cartier divisor on X and $f : X \to U$ a proper surjective morphism of normal varieties. We assume that*

(1) *$H|_S$ is base point free over U where $S = \lfloor \Delta \rfloor$,*

(2) *H is nef over U and there exists an integer $N > 0$ such that for any $n \geq N$, $nH - (K_X + \Delta)$ is nef and abundant over U.*

Then H is semi-ample over U.

Proof. See [Kawamata85], [Ambro05], [Fujino10], [Fujino12], [FG11] and [HX13, 4.1]. □

4.4. **Proof of (1.4).**

Proof of (1.4). After running a MMP with scaling over S and replacing S by a dominating generic finite base change, we may assume that $K_{X_\eta} + \Delta_\eta$ is nef over S by (3.6). Let $\Gamma = \lfloor \Delta \rfloor$, $\Theta = \Gamma - \lfloor \Delta \rfloor$ and G be the normalization of an irreducible component of Γ. Denote by $K_G + \Delta_G = (K_X + \Delta)|_G$, then (G, Δ_G) is dlt and by induction on the dimension, we know that $K_G + \Delta_G$ is semiample. By (4.5), we may assume that $K_{\Gamma_\eta} + \mathrm{Diff}^*_{\Gamma_\eta}\Theta_\eta$ is semiample. Thus, after shrinking S, we may assume that $K_\Gamma + \mathrm{Diff}^*_\Gamma\Theta$ is semiample over S. Let $\overline{k(\eta)}$ be the algebraic closure of $k(\eta)$ and $(X_{\bar\eta}, \Delta_{\bar\eta}) = (X_\eta, \Delta_\eta) \times_{k(\eta)} \overline{k(\eta)}$ the pair given by base change. By (4.6), it follows easily that $K_{X_{\bar\eta}} + \Delta_{\bar\eta}$ is semiample and hence so is $K_{X_\eta} + \Delta_\eta$. After possibly replacing S by a dense open subset, $K_X + \Delta$ is semi-ample over S. □

5. Proof of (1.2)

It is well known that (1.2), follows from the result below.

Theorem 5.1. *Fix a positive constant C and $n \in \mathbb{N}$ and a DCC set $\mathcal{A} \subset [0, 1]$, then there exists a constant N (which only depends on n, C and $\mathcal{A}$) such that if $(V, G = \sum g_i G_i)$ is an n-dimensional projective slc pair such that $K_V + G$ is ample, $(K_V + G)^n = C$ and $g_i \in \mathcal{A}$, then $N(K_V + G)$ is very ample.*

5.1. **Log canonical case.**

Proposition 5.2. $(5.1)_n$ *is true for normal pairs (V, G).*

Proof. We will prove the proposition by contradiction. Let us assume that there is an infinite sequence $\Psi = \{(V_j, G_j)\}$ as above, where the smallest integer $N_j > 0$ such that $N_j(K_{V_j} + G_j)$ is very ample satisfies $N_j \geq j$. Thus we can assume that any infinite subsequence of Ψ is also unbounded.

By [HMX12, Theorem C] and [HMX13, 3.1], $\{(V_j, G_j)\}_{j \in \mathbb{N}}$ forms a log birationally bounded family. In other words, we may assume that there exists a smooth variety S and a family of simple normal crossing pairs $(\mathcal{Y}, \mathrm{Supp}(\mathcal{D}))/S$ such that if we let $p_j : X_j \to V_j$ be a sufficiently high log resolution of (V_j, G_j) and define

$$\Delta_j = (p_j^{-1})_* G_j \cup \mathrm{Exc}(p_j),$$

then the pair (X_j, Δ_j) admits a birational morphism $q_j : X_j \to Y_j$ to a fiber $(Y_j, D_j) := (\mathcal{Y}, \mathrm{Supp}(\mathcal{D})) \times_S \{t_j\}$ with the property that

$$q_{j,*}\mathrm{Supp}(\Delta_j) \subset \mathrm{Supp}(D_j), \qquad \text{where } D_j = \mathcal{D}|_{Y_j}$$

After a possible étale base change of S, we can assume the irreducible components $\mathcal{D}^i$ of $\mathcal{D}$ intersect each fiber Y_s in an irreducible component $\mathcal{D}^i_s$,

Furthermore, by [HMX13, Proof of 5.1], there exists divisors $\Delta'_j \leq \Delta_j$ such that if we write

$$p_*\Delta'_j = \sum_i a^i_j D^i_j, \qquad \sum_i a^i \mathcal{D}^i = \mathcal{D}$$

for any fixed i, where the coefficients a^i_j are non-decreasing with limit a^i, then for all $j \in \mathbb{N}$, (X_j, Δ_j) and (X_j, Δ'_j) have the same pluri log canonical sections (i.e. $H^0(m(K_{X_j} + \Delta_j)) = H^0(m(K_{X_j} + \Delta'_j))$ for any $m \in \mathbb{N}$).

The above situation is summarized by the following diagram.

$$\begin{tikzcd}
 & (X_j, \Delta'_j \leq \Delta_j) \arrow[dl, "p_j"'] \arrow[dr, "q_j"] & \\
(V_j, G_j) & & (Y_j, D_j) \in (\mathcal{Y}, \mathcal{D})/S
\end{tikzcd}$$

By [HMX13, Section 5], for any $k \in \mathbb{N}$,

$$\lim_j \operatorname{vol}(K_{X_j} + \Delta'_j) = \operatorname{vol}(K_{Y_k} + D_k) = C.$$

We will need the following.

Lemma 5.3. *Let $f : (X, \Delta) \to (Y, D)$ be a birational morphism between log canonical pairs, with $f_*\Delta \leq D$. Assume that X is $\mathbb{Q}$-factorial and (X, Δ) is of log general type and has a log canonical model V. If*

$$\operatorname{vol}(K_Y + D) = \operatorname{vol}(K_X + \Delta),$$

then (X, Δ) and (Y, D) have the same log canonical model.

Proof. Replacing (X, Δ) by an appropriate resolution we can assume that X is smooth and there is a morphism $p : X \to V$. Replacing (Y, D) by the pair $(X, f^{-1}_* D + \operatorname{Exc}(f))$, we may assume $X = Y$. Let $\Delta_V = p_*\Delta$, then $H = p^*(K_V + \Delta_V)$ is big and nef. Since $\Delta \leq D$, it suffices to show that $p_*D = \Delta_V$. For $t \in [0, 1]$, we have

$$\operatorname{vol}(K_X+\Delta) = \operatorname{vol}(K_X+\Delta+t(D-\Delta)) \geq \operatorname{vol}(H+t(D-\Delta)) \geq \operatorname{vol}(H) = \operatorname{vol}(K_X+\Delta).$$

Therefore, $\operatorname{vol}(K_X + \Delta + t(D - \Delta)) = \operatorname{vol}(K_X + \Delta)$, is a constant function of t. Write $E = D - \Delta$. It follows from [LM09] that

$$0 = \frac{d}{dt}\operatorname{vol}(H + tE)|_{t=0} = n \cdot \operatorname{vol}_{X|E}(H) \geq n \cdot E \cdot H^{n-1} = n \cdot \deg p_*E.$$

Therefore $p_*E = 0$. $\square$

Since (V_j, G_j) is the log canonical model of (X_j, Δ'_j), by the above lemma, (V_j, G_j) is also a log canonical model of (Y_j, D_j). In particular, the coefficients of $G_j = p_* D_j$ are contained in a finite set.

After replacing S by a subvariety, we may assume that $\{t_j\}$ is a dense subset of S. By taking a log resolution and shrinking the base S, we can assume that $(\mathcal{Y}, \mathcal{D})$ is log smooth over S. Let $\nu_j : Y'_j \to Y_j$ be a log resolution such that $\mu_j : Y'_j \to V_j$ is a birational morphism and let $K_{Y'_j} + D'_j = \nu_j^*(K_{Y_j} + D_j) + E_j$ where D'_j and E_j are effective and have no common components. By the proof of [HX13, 2.12], (Y'_j, D'_j) has a good minimal model over V_i which is easily seen to be a good minimal model. Then it follows from [HX13, 2.10] that (Y_j, D_j) has a good minimal model as well.

It follows from (1.4) that there exists an open set $S^0 \subset S$ such that the relative log canonical model $(\mathcal{V}, \mathcal{G})$ of $(\mathcal{Y}, \mathcal{D})$ over S^0 exists.

The fibers of $(\mathcal{V}, \mathcal{G})$ over t_j are isomorphic to (V_j, G_j), which contradicts the assumption that $\{(V_j, G_j)\}$ is not contained in any bounded family. This finishes the proof of (5.2). $\square$

5.2. **The general case.** We will now prove the general case of (5.1). By contradiction, let us assume that there exists a sequence of (V_j, G_j) of slc pairs as in (5.1), such that $\lim_j N_j = \infty$, where N_j is the smallest positive integer, such that $N_j(K_{V_j} + G_j)$ is a very ample Cartier divisor.

Let us first quote the following theorem in [HMX12], which was conjectured by Alexeev and Kollár.

Theorem 5.4 ([HMX12, 1.3])**.** *Fix a positive integer n and a DCC set $\mathcal{A} \subset [0,1]$. Then the set*

$$\mathcal{V}(\mathcal{A}, n) = \{\mathrm{vol}(K_X + \Delta) | (X, \Delta) \text{ is lc, } \dim X = n \text{ and } \Delta \in \mathcal{A}\}$$

satisfies the DCC condition. In particular, the set $\mathcal{V} - \{0\}$ has a positive minimum.

As mentioned in the introduction, the datum of a canonically polarized slc pair (V, G) is equivalent to that of a quadruple

$$(\bar{V}, \bar{G}, \bar{D}, \tau)$$

satisfying the conditions given there (see [Kollár11] or [Kollár13b, Section 5]). We have

- for every s, let G^s be the restriction of $\bar{G} + \bar{D}$ to V^s, then (V^s, G^s) is log canonical and $K_{V^s} + G^s$ is ample;
- $\bar{G} + \bar{D} \in \mathcal{A} \cup \{1\}$;
- $(K_V + G)^n = (K_{\bar{V}} + \bar{G} + \bar{D})^n = \sum_s (K_{V^s} + G^s)^n$.

By (5.4), the set $\{\mathrm{vol}(K_{V^s} + G^s)\}$ satisfies the DCC. Thus there are only finitely many ways to write $C = (K_V + G)^n$ as a sum of elements of this set. In particular we may assume that the number of components V^s of $\bar{V}$ is fixed and that for any s and the quantity $(K_{V^s}+G^s)^n$ is fixed.

By (5.1), the data $(\bar{V},\bar{G},\bar{D})$ is bounded. In particular, we can assume that there is a fixed number N, such that $N(K_{\bar{V}}+\bar{G}+\bar{D})$ gives a very ample Cartier divisor $\bar{H}$ on $\bar{V}$.

Let $\tau : \bar{D}^n \to \bar{D}^n$ be the involution on the bounded family $\bar{D}^n$, the graph of τ in $\bar{D}^n \times \bar{D}^n$ belongs to an open subset of components of bounded degree of $\mathrm{Hilb}(\bar{D}^n \times \bar{D}^n)$ (parametrizing subschemes of degree $\leq 2^{n-1}\bar{D}^n \cdot H^{n-1}$ with respect to the ample bundle $H \boxtimes H$). Here we use the fact that

$$\tau^*(K_{\bar{D}^n} + \mathrm{Diff}_{\bar{D}^n}(\bar{G})) = K_{\bar{D}^n} + \mathrm{Diff}_{\bar{D}^n}(\bar{G}).$$

Since $\bar{D}^n \cdot H^{n-1}$ is bounded, it follows that the 'glueing data' τ is also bounded.

Therefore, after base change, we can assume that there is a variety S of finite type and a dense set of points $\{s_j\} \in S$, such that

(1) There exists a flat family of (possibly non-connected) log canonical pairs $(\bar{\mathcal{V}}, \bar{\mathcal{D}}+\bar{\mathcal{G}})$ over S with $K_{\bar{\mathcal{V}}}+\bar{\mathcal{D}}+\bar{\mathcal{G}}$ relatively ample,
(2) $(\bar{V}_j, \bar{D}_j+\bar{G}_j) \cong (\bar{\mathcal{V}}, \bar{\mathcal{D}}+\bar{\mathcal{G}}) \times_S \{s_j\}$.

As $\mathrm{Aut}_S(\bar{\mathcal{D}}^n)$ is an open subscheme of the Hilbert scheme $\mathrm{Hilb}(\bar{\mathcal{D}}^n \times_S \bar{\mathcal{D}}^n/S)$, and all the points $[\Gamma_{\tau_j}]$ corresponding to the graphs of involution of $(\bar{D}^n, \mathrm{Diff}_{\bar{D}^n}(\bar{G}))$ are of bounded degree, after choosing a subsequence of $\{s_j\}$ and replacing S by the closure of $\{s_j\}$, we can assume that we can take the closure of $\{\tau_j\}$ in $\mathrm{Aut}_S(\bar{\mathcal{D}}^n)$ to obtain a subvariety $\Xi \subset \mathrm{Aut}_S(\bar{\mathcal{D}}^n)$ dominating S. The condition that τ is an involution that preserves $\mathrm{Diff}_{\bar{D}^n}(\bar{G})$ is closed in $\mathrm{Aut}_S(\bar{\mathcal{D}}^n)$, and it holds over a dense set of points s_j of Ξ. Therefore any point in Ξ parametrizes an involution that preserves $\mathrm{Diff}_{\bar{D}^n}(\bar{G})$.

Applying the base change $\Xi \to S$ (and hence replacing S by Ξ), we obtain a quadruple $(\bar{\mathcal{V}}, \bar{\mathcal{D}}, \bar{\mathcal{G}}, \tau_S)$ over S which satisfies the family version of the 'glueing condition' in the introduction, i.e.

(1) $(\bar{\mathcal{V}}, \bar{\mathcal{D}}+\bar{\mathcal{G}})$ is a log canonical and proper over S,
(2) $K_{\bar{\mathcal{V}}}+\bar{\mathcal{D}}+\bar{\mathcal{G}}$ is ample over S, and
(3) $\Xi : \bar{\mathcal{D}}^n \to \bar{\mathcal{D}}^n$ is an involution on the normalization $\bar{\mathcal{D}}^n$ of $\bar{\mathcal{D}}$ and $\mathrm{Diff}_{\bar{\mathcal{D}}^n}\bar{\mathcal{G}}$ is Ξ-invariant.

Therefore by [Kollár11, 24], we conclude that $(\bar{\mathcal{V}}, \bar{\mathcal{D}}, \bar{\mathcal{G}}, \tau_S)$ is given by normalization of a slc pair $(\mathcal{V}, \mathcal{G})$ over S such that $K_{\mathcal{V}}+\mathcal{G}$ is ample over S. Therefore, there is a uniform N such that $N(K_{\mathcal{V}}+\mathcal{G})$ is very ample over S, which contradicts our assumptions. This concludes the proof.

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