THE CANONICAL RING IS FINITELY GENERATED

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1. Introduction

Let X be a smooth complex projective variety so that X is a subset of $\mathbb{P}^N$ cut out by finitely many homogeneous polynomials $P_i \in \mathbb{C}[z_0, \ldots, z_N]$. The canonical bundle of X is denoted by ω_X so that for all $m \geq 0$ sections $s \in H^0(X, \omega_X^{\otimes m})$ may be locally written as $f \cdot (dx_1 \wedge \ldots \wedge dx_n)^{\otimes m}$ where f is a holomorphic function and $x_1, \ldots, x_n$ are local parameters on X. The vector spaces $H^0(X, \omega_X^{\otimes m})$ give rise to the canonical ring

$$R(\omega_X) := \bigoplus_{m \geq 0} H^0(X, \omega_X^{\otimes m}).$$

This ring is of fundamental importance in the study of the birational geometry of higher dimensional varieties. Recall that if X and X' are birational (i.e. they have isomorphic open subsets) then $H^0(X, \omega_X^{\otimes m}) \cong H^0(X', \omega_{X'}^{\otimes m})$. In particular $R(\omega_X)$ is a birational invariant of X. The purpose of this note is to give an overview of recent results in higher dimensional birational algebraic geometry that lead to the proof of the following:

Theorem 1.1. *Let X be a smooth complex projective variety. Then the canonical ring $R(\omega_X)$ is finitely generated.*

It should be noted that there are two announced proofs of this result. We will illustrate the approach of [HM05] and [BCHM06] which is based on the ideas of the minimal model program and in particular on ideas of V. Shokurov [Shokurov03]. This approach uses the methods of higher dimensional birational geometry and has the pleasant feature that it also allows us to prove many important results on the birational geometry of higher dimensional complex projective varieties such as the existence of flips and (under favorable, but not too restrictive conditions) the termination of certain sequences of flips and hence the existence of minimal models.

As mentioned above, there is another proof due to Y.-T. Siu [Siu06]. This is completely independent and is based on analytic methods. In this paper we will not discuss any of the details of the analytic approach.

We now recall some notation. $\mathbb{P}^n$ will denote n-dimensional complex projective space, i.e. a compactification of $\mathbb{C}^n$ obtained by adding a hyperplane (a copy of $\mathbb{P}^{n-1}$). A complex projective variety in $\mathbb{P}^n$ is given by the common zeroes of a finite set of homogeneous polynomials $P_1, \ldots, P_r \in \mathbb{C}[z_0, \ldots, z_n]$. Given a line bundle L on a variety X, $H^0(X, L)$ denotes the complex vector space of global sections of L. In particular $\mathcal{O}_{\mathbb{P}^n}(1)$ denotes the hyperplane line bundle so that $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r))$ may be identified with the homogeneous polynomials in $\mathbb{C}[z_0, \ldots, z_n]$ of degree r. There is a rational map $\phi_L : X \dashrightarrow \mathbb{P}^N = \mathbb{P}H^0(X, L)$ defined as follows: let $s_0, \ldots, s_N$ be a basis of $H^0(X, L)$ and $x \in X$, then $\phi_L(x) = [s_0(x), \ldots, s_N(x)]$. Note that ϕ_L is undefined at any point in the base locus of L.

For any line bundle L we let $R(L)$ be the graded ring given by $\bigoplus_{m \geq 0} H^0(X, L^{\otimes m})$. We remark that if k is a positive integer, then the ring $R(L)$ is finitely generated if and only if so is $R(L^{\otimes k})$ The number $\kappa(L) :=$ tr. deg. $R(L) - 1$ is the *Kodaira dimension* of X.

For any line bundle L and any curve $C \subset X$, we may define $L \cdot C = \deg(L|_C)$, and for any morphism $f : Y \to X$ we have a line bundle on Y given by f^*L. Similarly definitions also hold for any formal linear combination of line bundles $L = \sum r_i L_i$ where $r_i \in \mathbb{R}$.

For any birational morphism $f : Y \to X$, $\mathrm{Exc}(f)$ denotes the exceptional set, so that $Y - \mathrm{Exc}(f)$ is the biggest open subset of Y where f restricts to an isomorphism.

2. Geometry of curves

Curves are smooth complex projective varieties of dimension 1 also known as Riemann surfaces. These are topologically classified by their genus $g := \dim H^0(X, \omega_X)$ (recall that in this case $\omega_X = \Omega^1_X = T^\vee_X$ is the cotangent bundle and so g is just the number of linearly independent global holomorphic 1-forms). We can divide surfaces in to three rough classes:

1. $g = 0$. In this case there is only one possibility: $\mathbb{P}^1$. We have that $\omega_X = \mathcal{O}_{\mathbb{P}^1}(-2)$ so that the elements of $H^0(\mathbb{P}^1, \omega_X^{\otimes m}) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2m))$ correspond to homogeneous polynomials of degree $-2m$ and hence they are all 0. In particular $R(X) \cong \mathbb{C}$.
2. $g = 1$. In this case there is a 1-parameter family of elliptic curves. We have that $\omega_X = \mathcal{O}_X$ and so $R(X) \cong \mathbb{C}[t]$.
3. $g \geq 2$. In this case there is a $(3g - 3)$-parameter family of such curves. One can consider sections of the line bundle $L :=$

$\omega_X^{\otimes 3}$. It is known that the sections of L define an embedding $\phi_L : X \to \mathbb{P}^{5g-6} = \mathbb{P}H^0(L)$. One has that $H^0(X, \omega_X^{\otimes 3m}) \cong H^0(X, \mathcal{O}_{\mathbb{P}^{5g-6}}(m)|_X)$. For any $m \gg 0$ the homomorphisms $H^0(\mathbb{P}^{5g-6}, \mathcal{O}_{\mathbb{P}^{5g-6}}(m)) \to H^0(X, \mathcal{O}_{\mathbb{P}^{5g-6}}(m)|_X)$ are surjective. Since $R(\mathcal{O}_{\mathbb{P}^{5g-6}}(1))$ is finitely generated, then so is $R(\omega_X)$.

The natural question is if one can then generalize these results to higher dimensions.

3. Geometry of Surfaces

The first problem that one encounters in the classification of surfaces (i.e. complex projective varieties of dimension 2) is that given any surface X, one can produce a new surface by blowing up a point $x \in X$. This produces a morphism $f : X' = \mathrm{bl}_x X \to X$ which is an isomorphism over $X - x$ and replaces the point x by a -1 curve i.e. a rational curve $E \cong \mathbb{P}^1$ such that $E^2 = -1$. The points of E correspond to the tangent directions at x. It is then reasonable to attempt to classify surfaces modulo birational isomorphism, so that two surfaces (or higher dimensional varieties) are equivalent if they have isomorphic open subsets. It is known that two surfaces are birational if they are isomorphic after a finite sequence of blow ups. By Castelnuovo's criterion one may blow down any -1 curve, so that one sees that any surface is birational to a minimal surface i.e. a surface that contains no -1 curves.

The second problem is what invariant should replace the genus in higher dimension. One option would be to take the topological type of X, but this notion is too rigid and in particular it is not invariant under birational maps. It turns out that a better choice is given by the vector spaces $H^0(X, \omega_X^{\otimes m})$ for $m > 0$. An element of $H^0(X, \omega_X^{\otimes m})$ is called a global m-th pluricanonical form. It can be locally written as $f \cdot (dz_1 \wedge dz_2)^{\otimes m}$ where f is a holomorphic function and z_1 and z_2 are local parameters. As mentioned above, we have a graded ring known as the canonical ring $R(\omega_X) = \bigoplus_{m>0} H^0(X, \omega_X^{\otimes m})$ which is a birational invariant of X. One may also define a coarser invariant, known as the Kodaira dimension of X given by

$$\kappa(X) = \text{tr. deg.} R(\omega_X) - 1 \in \{-1, 0, 1, \dots, \dim X\}.$$

We say that X is of general type if $\kappa(X) = \dim(X)$. For surfaces, we then have $\kappa(X) \in \{-1, 0, 1, 2\}$.

The Enriques-Iitaka classification of surfaces may be described as follows.

(1) $\kappa(X) = -1$: X is covered by rational curves (in fact X is birational to $C \times \mathbb{P}^1$ for some curve C of genus $g(C) = \dim H^0(\Omega^1_X)$). Therefore $H^0(X, \omega_X^{\otimes m}) \cong 0$ for all $m > 0$ and so the canonical ring $R(\omega_X)$ is isomorphic to $\mathbb{C}$.

   Note that if $\kappa(X) = -1$, then X has many different minimal surfaces (but their relationships are well understood). On the other hand, if $\kappa(X) \geq 0$, it is known that X has a unique minimal surface say X'.

(2) $\kappa(X) = 0$: X is birational to a unique minimal surface X' which is in one of four well understood classes of surfaces (abelian, K3, Enriques and bielliptic surfaces). One has that $\omega_{X'}^{\otimes 12} \cong \mathcal{O}_{X'}$. The canonical ring $R(\omega_X)$ is isomorphic to $\mathbb{C}[t]$.

(3) $\kappa(X) = 1$: X is covered by elliptic curves. In fact X is birational to a unique minimal surface X' which admits a morphism $f : X' \to C$ such that $\omega_{X'}^{\otimes 12} = f^*L$ for some line bundle L of positive degree on C. It then follows that the canonical ring $R(\omega_X)$ is finitely generated since $R(\omega_X^{\otimes 12}) \cong R(L)$ is finitely generated.

(4) $\kappa(X) = 2$: X is birational to a unique minimal surface X' such that $\omega_{X'}$ is nef (i.e. $\deg(\omega_{X'}|_C) \geq 0$ for any curve $C \subset X'$). $H^0(\omega_{X'}^{\otimes 5})$ defines a birational morphism $f : X' \to \mathbb{P}^N = \mathbb{P}H^0(\omega_{X'}^{\otimes 5})$ which contracts all rational curves $E \cong \mathbb{P}^1$ with $E^2 = -2$ to a rational double point singularity. We therefore have that $\omega_{X'}^{\otimes 5} \cong f^*\mathcal{O}_{\mathbb{P}^N}(1)$ so that $R(\omega_X)$ is finitely generated and $f(X') \cong \mathrm{Proj} R(\omega_X)$.

4. Geometry of Threefolds

One would like to generalize the above classification to the case of threefolds. This is possible but there are several new features which made the problem extremely difficult. The classification was achieved by work of Kawamata, Kollár, Mori, Reid, Shokurov and others which culminated in Mori's construction of flips [Mori88]. The upshot is the following.

Theorem 4.1. *Let X be a smooth complex projective 3-fold.*

(1) *If $\kappa(X) = -1$, then X is covered by rational curves.*
(2) *If $\kappa(X) \geq 0$, then X has a minimal model.*

In all cases the canonical ring $R(\omega_X)$ is finitely generated.

It is important to notice that the minimal model is not unique and may have mild singularities. We must in fact allow terminal singularities. These are mild singularities in particular they are rational singularities that occur in codimension ≥ 3 and by definition

$\omega_X^{\otimes m}$ is a line bundle for some $m > 0$ so that we may still define $\omega_X \cdot C = \frac{1}{m}\deg(\omega_X^{\otimes m}|_C)$. In dimension 3 these singularities are classified, but in higher dimension they are somewhat more mysterious (but still well behaved).

There is also an explicit procedure for constructing a minimal model known as the minimal model program. To run a minimal model program, one starts with a terminal complex projective variety X. If ω_X is nef, then we are done. Otherwise, let

$$\mathrm{N}_1(X) = \{\sum c_i C_i | c_i \in \mathbb{R},\ C_i \text{ is a curve in } X\}/\equiv$$

where $C \equiv D$ (that is the curves C and D are numerically equivalent) if $(C - D) \cdot L = 0$ for any line bundle L on X. We let $\rho(X) = \dim_{\mathbb{R}}(\mathrm{N}_1(X))$. Let $\overline{\mathrm{NE}}(X)$ be the closure of the quotient of the cone of effective cones on X. If ω_X is not nef, then by the Cone Theorem there is an ample line bundle A and a rational number a such that $\omega_X \otimes A^{\otimes a}$ is nef and a unique *negative extremal ray* $R = \mathbb{R}^+[C]$ where C is curve in X and $\omega_X \cdot C' = 0$ if and only if $[C'] \in R$. By the Base Point Free Theorem, there is then a morphism $f : X \to Z$ (surjective with connected fibers) such that for any curve $D \subset X$, $f_* D = 0$ if and only if $[D] \in R$. There are several cases to consider.

(1) If $\dim(Z) < \dim(X)$, then $f : X \to Z$ is called a *Mori fiber space*. It has the following properties: $\rho(X) - \rho(Z) = 1$; $\omega_X \cdot C < 0$ for any curve contracted by f; the fibers of f are covered by rational curves (in fact rationally connected). This gives a clear geometric reason why in this case $\kappa(X) = -1$ as ω_X has negative degree on this covering family of rational curves and so any element of $H^0(X, \omega_X^{\otimes m})$ must vanish on this covering family of rational curves and hence must be 0.

(2) If $\dim(Z) = \dim(X)$ and $\dim(\mathrm{Exc}(f)) = \dim(Z) - 1$, then f is a *divisorial contraction*. In this case Z also has terminal singularities and we may simply replace X by Z. This is the analog of the contraction of a -1 curve in the surface case. We have $\rho(Z) = \rho(X) - 1$ and hence this process can be repeated only finitely many times.

(3) If $\dim(Z) = \dim(X)$ and $\dim(\mathrm{Exc}(f)) < \dim(Z) - 1$, then f is a *small contraction*. In this case Z does not have terminal singularities. In fact $\omega_X^{\otimes m}$ is not a line bundle for any $m > 0$ and so one cannot make sense of the intersection number $\omega_X \cdot C$ of some curves $C \subset X$. The (very bold!) solution is then to replace X by its flip. The flip of $f : X \to Z$ is a morphism $f^+ : X^+ \to Z$ such that X is isomorphic to X^+ outside a codimension ≥ 2

subset, $\rho(X)=\rho(X^+)=\rho(Z)+1$ and $\omega_{X^+}\cdot C>0$ for any curve $C\subset X^+$ contracted by f^+. Therefore a flip can be thought of a codimension 2 surgery which replaces some ω_X negative curves by ω_{X^+} positive curves. Since our goal is to arrive to a minimal model (or to a Mori fiber space), this would seem to be a step in the right direction. The good news is that if the flip exists, then it is uniquely defined by the formula

$$X^+ = \mathrm{Proj}_Z \bigoplus_{m\geq 0} f_*(\omega_X^{\otimes m})$$

and it has mild singularities. There are two items of bad news:

First of all it is very hard to prove the existence of X^+. In fact (assuming that Z is affine) this is equivalent to showing that $R(\omega_X)$ is finitely generated as an $\mathcal{O}_Z$ module. At first glance this would seem hopeless as it is equivalent to one of the original motivating problems: to show that $R(\omega_X)$ is finitely generated. Upon further reflection, one notices that as we want to only show that $R(\omega_X)$ is finitely generated over Z, the problem might be more accessible, especially in view of the fact that $\dim(X)=\dim(Z)$. At any rate, Mori solved the problem in the most geometric (but hardest) possible way: he classified all possible flipping contractions $f:X\to Z$ of this type and then constructed the corresponding flip X^+.

Secondly, one must show that any sequence of flips terminates. Luckily in dimension 3 this is not too difficult.

5. Minimal Model Program for log pairs

The Minimal Model Program is expected to work in the more general setting of log pairs. We let K_X denote a canonical divisor i.e. a divisor corresponding to ω_X. A *log pair* (X,B) consists of a normal variety X and a $\mathbb{R}$-divisor $B=\sum b_iB_i$ (i.e. $b_i\in\mathbb{R}$ and B_i are irreducible codimension 1 subvarieties) and K_X+B is $\mathbb{R}$-Cartier (so that you may think of it as a formal linear combination of line bundles with real coefficients). Therefore, it still makes sense to consider pull-backs $f^*(K_X+B)$ and to intersect K_X+B with curves $C\subset X$. In particular we can ask the question *is* (X,B) *a minimal model?* i.e. is $(K_X+B)\cdot C\geq 0$ for any curve $C\subset X$.

Just as in the case with no boundary (i.e. $B=0$), we hope to find a birational map $\phi: X \dashrightarrow Y$ consisting of flips and divisorial contractions such that either

(1) (Y,ϕ_*B) is a minimal model (i.e. $K_Y+\phi_*B$ is nef), or

(2) $(Y, \phi_* B)$ is a Mori fiber space (i.e. there is a surjective morphism with connected fibers $f : Y \to S$ such that $\rho(Y) - \rho(S) = 1$ and $-(K_Y + \phi_* B) \cdot C < 0$ for any curve C contracted by f).

To achieve this we must require that the pair (X, B) have mild singularities. It turns out that one should require that (X, B) have kawamata log terminal singularities, so that if we have $f : X' \to X$ a morphism from a smooth variety X' such that the components of the transform of B and of the exceptional divisor are smooth and transverse, then we may write $f^*(K_X + B) = K_{X'} + B'$ where $B' = \sum b'_i B'_i$, $f_* B' = B$ and $b'_i < 1$ for all i.

The Cone Theorem is known to hold for kawamata log terminal pairs in any dimension. Therefore the main questions to answer are:

Question 5.1. *Let (X, B) be a kawamata log terminal pair.*

(1) *Do flips exists?*
(2) *Is any given sequence of flips finite?*

A positive answer to these questions would then allow us to construct minimal models in all dimension and hence show that pluricanonical rings are finitely generated.

6. Higher dimensional varieties

Using ideas of Shokurov, in [BCHM06] it is shown that:

Theorem 6.1. *Let (X, Δ) be a kawamata log terminal pair such that one of the following holds:*

(1) $\kappa(K_X + B) = \dim X$, *or*
(2) $\kappa(B) = \dim X$, *or*
(3) *(X, B) is not pseudo-effective, (i.e. for some ample divisor A and some $0 < \epsilon \ll 1$, we have $\kappa(K_X + B + \epsilon A) = -1$).*

Then there exists a finite sequence of flips and divisorial contractions $\phi : X \dashrightarrow Y$ such that $(Y, \phi_* B)$ is either a minimal model or is a Mori fiber space.

Corollary 6.2. *If X is a smooth complex projective variety, then its canonical ring $R(K_X)$ is finitely generated.*

Proof. This follows from a result of Mori and Fujino [FM00] according to which it suffices to prove finite generation for any kawamata log terminal pair (Y, B) with $\kappa(K_Y + B) = \dim Y$. □

We remark that we prove that flips of kawamata log terminal pairs exist in all dimensions but we do not show that sequences of flips terminate. What we show is that under the right hypothesis, there exists a carefully chosen sequence of flips that terminates. These sequences of flips are know as flips with scaling. To explain this, suppose that we have a kawamata log terminal pair (X, B), then we may choose an ample divisor A and a number $0 < \tau_0 \leq 1$ such that $K_X + B + \tau_0 A$ is ample (and hence nef). Then we let

$$\tau_1 := \inf\{t > 0 | K_X + B + tA \text{ is nef}\}.$$

If $\tau_1 = 0$, we are done as $K_X + B$ is nef. Otherwise there is a $K_X + B$ negative extremal ray $R = \mathbb{R}^+[C]$ such that $(K_X+B+\tau_1 A)\cdot C = 0$. If R induces a Mori fiber space, we are also done. Otherwise, we perform the corresponding flip or divisorial contraction say $\phi_1 : X \dashrightarrow X_1$. Since $(K_X+B+\tau_1 A)\cdot C = 0$ it follows that also $K_{X_1}+(\phi_1)_*B+\tau_1(\phi_1)_*A$ is nef. We may therefore repeat this procedure. Proceeding in this way, we obtain a sequence of flips and divisorial contractions $\phi_i : X_{i-1} \dashrightarrow X_i$ and numbers $0 \leq \tau_{i-1} \leq \tau_i \leq 1$ such that $K_{X_i} + B_i + \tau_i A_i$ is nef. This sequence ends if $\tau_n = 0$ or if we obtain a Mori fiber space. Otherwise we have an infinite sequence of minimal models $(X_i, B_i + \tau_i A_i)$.

The key idea is that if $\kappa(B) = \dim X$, then by using a compactness argument, we can show that there are only finitely many distinct minimal models for $(X, B + tA)$ where $0 \leq t \leq 1$.

In order to show that flips exist, we use Shokurov's so called reduction to PL-flips. The PL here stands for pre-limiting. Recall that given a flipping contraction $f : X \to Z$, to construct the flip of X, it suffices to show that

$$R(K_X + B) = \bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(m(K_X + B)))$$

is finitely generated as an $\mathcal{O}_Z$ algebra (here we are assuming for simplicity that Z is affine). The main idea is that after the reduction to PL-flips, we may assume that $B = B_0 + \sum b_i B_i$ where $b_i \in \mathbb{Q}$ and $0 \leq b_i < 1$ (more precisely the pair (X, B) is purely log terminal so that if $\nu : X' \to X$ is a birational map and $K_{X'} + B' = \nu^*(K_X + B)$, then all coefficients of B' are < 1 except for the coefficient of the transform of B_0 which equals 1). We may also assume that a further technical condition holds, namely that for some positive number q, we have $(K_X + B - qB_0)\cdot C = 0$ for all curves $C \subset X$ contracted by $f : X \to Z$.

We then let $S := B_0$ and we look at the restriction map

$$\Psi : \bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(m(K_X + B))) \to \bigoplus_{m\geq 0} H^0(S, \mathcal{O}_S(m(K_S + B_S))).$$

From the definition of purely log terminal singularity, one can see that the pair (S, B_S) is kawamata log terminal. The idea is that the kernel of this map of graded rings is essentially a principal ideal so that in order to show that $R(K_X+B)$ is finitely generated, it suffices to show that $\mathrm{Im}(\Psi)$ is finitely generated. Therefore if Ψ is surjective, we can then conclude by induction on the dimension. Unluckily this is too much to expect. However, in [HM05], we show that after replacing X and S by suitable birational models, there is a divisor $0 \leq \Theta \leq B_S$ such that

$$\mathrm{Im}(\Psi) = \bigoplus_{m\geq 0} H^0(S, \mathcal{O}_S(m(K_S+\Theta))).$$

Note that (S,Θ) is also a kawamata log terminal pair and so if the coefficients of Θ are rational we are once again done by induction on the dimension. It should be remarked however that Θ is obtained by a limiting procedure and hence is a priori only an $\mathbb{R}$-divisor. This problem can be addressed (as already observed by Shokurov) by using techniques of diophantine approximation.

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