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Boundedness Results in Birational Geometry

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Abstract

We survey results related to pluricanonical maps of complex projective varieties of general type.

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1. Introduction

Let X be a smooth complex projective variety of dimension n. In order to study the geometry of X one would like to choose a natural embedding $X \subset \mathbb{P}^N_{\mathbb{C}}$. This is equivalent to the choice of a very ample line bundle $\mathcal{L}$ on X i.e. of a line bundle $\mathcal{L}$ such that its sections define an embedding

$$\phi_{\mathcal{L}} : X \hookrightarrow \mathbb{P}H^0(X, \mathcal{L}).$$

(If $s_0, \ldots, s_N$ is a basis of $H^0(X, \mathcal{L})$, then we let $\phi_{\mathcal{L}}(x) = [s_0(x) : \ldots : s_N(x)]$.) Conversely, given an embedding $\phi : X \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$, we have that $\mathcal{L} := \phi^*\mathcal{O}_{\mathbb{P}^N}(1)$ is a very ample line bundle on X. Since any projective variety X may have many different embeddings in $\mathbb{P}^N$ it is important to find a "natural" choice of this embedding (or equivalently a natural choice of a very ample line bundle).

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The only known natural choice is the canonical bundle $\omega_X = \wedge^n T_X^\vee$ and its tensor powers $\omega_X^{\otimes m}$ for $m \in \mathbb{Z}$.

When $\dim X = 1$, we have that ω_X is a line bundle of degree $\deg \omega_X = 2g-2$ where g denotes the genus of X so that $\deg \omega_X > 0$ if and only if $g \geq 2$. There exist curves with genus $g \geq 2$ such that ω_X is not very ample, however we have the following classical result.

Theorem 1.1. *If X is a curve of genus $g \geq 2$, then $\omega_X^{\otimes m}$ is very ample for any integer $m \geq 3$.*

Proof. Let $\phi_m = \phi_{\omega_X^{\otimes m}}$. In order to show the theorem, we must show that ϕ_m is a morphism and separates points and tangent directions. This is equivalent to showing (cf. [Hartshorne77, II.7.3, IV.3.1]) that

1. $h^0(X, \omega_X^{\otimes m}(-P)) = h^0(X, \omega_X^{\otimes m}) - 1$ for any $P \in X$, and
2. $h^0(X, \omega_X^{\otimes m}(-P-Q)) = h^0(X, \omega_X^{\otimes m}) - 2$ for any points P and Q on X.

Considering the short exact sequence of coherent sheaves on X

$$0 \to \omega_X^{\otimes m}(-P) \to \omega_X^{\otimes m} \to \mathbb{C}_P \to 0$$

(where the last homomorphism is given by evaluating sections at P) we obtain a short exact sequence of vector spaces over $\mathbb{C}$

$$0 \to H^0(X, \omega_X^{\otimes m}(-P)) \to H^0(X, \omega_X^{\otimes m}) \to \mathbb{C} \to H^1(X, \omega_X^{\otimes m}(-P)) \ldots$$

Since $\deg \omega_X^{\otimes(1-m)}(P) = (1-m)(2g-2) - 1 < 0$, we have that

$$H^1(X, \omega_X^{\otimes m}(-P)) \cong H^0(X, \omega_X^{\otimes(1-m)}(P))^\vee = 0,$$

and so the homomorphism $H^0(X, \omega_X^{\otimes m}) \to \mathbb{C}$ is surjective (or equivalently $h^0(X, \omega_X^{\otimes m}(-P)) = h^0(X, \omega_X^{\otimes m}) - 1$). Therefore ϕ_m is a morphism.

The proof that ϕ_m separates points and tangent directions is similar. $\square$

Remark 1.2. *Note that:*

1. *If $\mathcal{L}$ is a line bundle, then $\mathcal{L}(-P)$ denotes the coherent sheaf of sections of $\mathcal{L}$ vanishing at P. Since $\dim X = 1$ this is also a line bundle.*
2. *$h^0(X, \mathcal{L})$ denotes the dimension of the $\mathbb{C}$-vector space $H^0(X, \mathcal{L})$.*
3. *The isomorphism $H^1(X, \omega_X^{\otimes m}(-P)) \cong H^0(X, \omega_X^{\otimes(1-m)}(P))^\vee$ is implied by Serre Duality: If X is a smooth projective variety of dimension n and F is locally free, then $H^i(X, F) \cong H^{n-i}(X, \omega_X \otimes F^\vee)^\vee$.*
4. *The vanishing $H^1(X, \omega_X^{\otimes m}(-P))$ is also implied by Kodaira vanishing which says that if X is a smooth projective variety and $\mathcal{L}$ is an ample line bundle, then $H^i(X, \omega_X \otimes \mathcal{L}) = 0$ for all $i > 0$.*

It follows that we can hope to use some *multiple* of ω_X to study the geometry of *most* varieties. In dimension ≥ 2 the situation is further complicated by the fact that there exist (smooth) birational varieties which are not isomorphic. For example if $P \in X$ is a point on a smooth projective surface (i.e. $\dim X = 2$), then one can construct a new surface $X' = \mathrm{Bl}_P X$ the *blow up* of X at P such that there is a morphism $f : X' \to X$ which is an isomorphism over the complement of P and whose fiber over P is a curve $E \subset X'$ which is isomorphic to $\mathbb{P}^1 \cong \mathbb{P}(T_x X)$. It is easy to see that $E \cdot E = -1$ and (since $\omega_{X'} = f^*\omega_X \otimes \mathcal{O}_X(E)$) that $\omega_{X'} \cdot E = -1$. Therefore, E is known as a -1-*curve.*

Consider now the example of a quintic surface $X \subset \mathbb{P}^3$ which is a smooth surface defined by a homogeneous polynomial of degree 5 in $\mathbb{C}[x_0, \ldots, x_3]$. By adjunction, one has $\omega_X \cong \omega_{\mathbb{P}^3}(X) \otimes \mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^3}(1)|_X$ so that ω_X is very ample (and $\phi_1 : X \hookrightarrow \mathbb{P}^3$ coincides with the given embedding). However, if $f : X' \to X$ is the blow up of X at a point $P \in X$ and E is the exceptional curve, then $\omega_{X'} \cdot E = -1$. Therefore, for any $m > 0$, sections of $H^0(X', \omega_{X'}^{\otimes m})$ must vanish along E and so ϕ_m is not a morphism along points of E. If we remove the singularities of ϕ_m (or more precisely we subtract the fixed divisor mE of $\omega_{X'}^{\otimes m}$), we obtain a morphism $\phi_{\omega_{X'}^{\otimes m}(-mE)} : X' \to \mathbb{P}^3$ whose image is X.

Therefore in dimension ≥ 2, we can not expect that, for most varieties, multiples of ω_X define an embedding in projective space. We can only hope that for most varieties, multiples of ω_X define a birational map (i.e. there is an open subset of X on which the given map is an embedding).

We have the following definition.

Definition 1.3. *Let X be a smooth projective variety, then X is of general type if the sections of $\omega_X^{\otimes m}$ define a birational map for some $m > 0$.*

It is known that if X is of general type, then in fact the sections of $\omega_X^{\otimes m}$ define a birational map for *all* sufficiently big integers $m > 0$.

When $\dim X = 2$ (and X is of general type), it is known by a result of Bombieri (cf. [Bombieri70]) that:

Theorem 1.4. *If X is a surface of general type, then ϕ_m is birational for all $m \geq 5$.*

In fact we have that (after subtracting the fixed divisor) $\phi_m : X \to \mathbb{P}^N$ is a morphism whose image X_{can} is uniquely determined by $X_{\mathrm{can}} \cong \mathrm{Proj}\, R(\omega_X)$ where $R(\omega_X) = \oplus_{m \geq 0} H^0(X, \omega_X^{\otimes m})$ is the canonical ring. Note that X_{can} has rational double point singularities so that $\omega_{X_{\mathrm{can}}}$ is a line bundle. We have $\omega_X = \phi_m^* \omega_{X_{\mathrm{can}}} \otimes \mathcal{O}_X(E)$ for some effective exceptional divisor E or equivalently $\omega_X^{\otimes m} \otimes \mathcal{O}_X(-mE) \cong \phi_m^* \mathcal{O}_{\mathbb{P}^N}(1)$.

Since X_{can} may be singular, it is convenient to consider the minimal desingularization $X_{\min} \to X_{\mathrm{can}}$. For surfaces of general type, the minimal model is uniquely determined. It can also be obtained from X by contracting all -1 curves. Therefore there is a morphism $X \to X_{\min}$. It is known that $\omega_{X_{\min}}^{\otimes m}$ is base point free for all $m \geq 5$ (in fact ϕ_m defines the morphism $X_{\min} \to X_{\mathrm{can}}$ and $\phi_m^* \omega_{X_{\mathrm{can}}} = \omega_{X_{\min}}$).

By Riemann-Roch and (a generalization of) Kodaira vanishing, we have that for all $m \geq 2$

$$h^0(\omega_X^{\otimes m}) = h^0(\omega_{X_{\min}}^{\otimes m}) = \frac{m(m-1)}{2} K_{X_{\min}}^2 + \chi(\mathcal{O}_{X_{\min}})$$

where $K_{X_{\min}}^2 \in \mathbb{Z}_{>0}$ is the self intersection of the canonical divisor $K_{X_{\min}}$ (a divisor corresponding to the zeroes of a section of $\omega_{X_{\min}}$). Note that as X is of general type

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X_{\min}}) = \sum (-1)^i h^i(\mathcal{O}_{X_{\min}}) > 0.$$

In particular we have that for all $m \geq 2$

$$P_m(X) := h^0(\omega_X^{\otimes m}) > \frac{m(m-1)}{2} K_{X_{\min}}^2 .$$

One important consequence of the above results is that X_{can} is a subvariety of $\mathbb{P}^{10K_{X_{\min}}^2 + \chi(\mathcal{O}_{X_{\min}})-1}$ of degree $25K_{X_{\min}}^2$. It follows by a Hilbert scheme type argument that there exists a parameter space for canonical (and hence also for minimal) surfaces of general type:

Theorem 1.5. *Let $M \in \mathbb{Z}_{>0}$. There exists a morphism $\mathcal{X} \to S$ such that for any $s \in S$, the fiber $\mathcal{X}_s$ is a canonical surface of general type and for any canonical surface of general type X such that $K_X^2 \leq M$, there is a point $s \in S$ and an isomorphism $X \cong \mathcal{X}_s$.*

Remark 1.6. *The moduli space for minimal complex projective surfaces of general type was constructed in [Gieseker77].*

It is then important to generalize (1.4) to higher dimensions. Even though many of the features of the classification of surfaces of general type were shown to hold for threefolds in the 80's (cf. [Kolláretal92]), the generalization of (1.4) turned out to be more difficult than expected and was only completed in [Tsuji07], [HM06] and [Takayama06]. One of the difficulties encountered, is that in dimension ≥ 3 even though minimal models $X_{\min}$ are known to exist (but are not uniquely determined cf. [BCHM09]), they have mild (terminal) singularities and so $K_{X_{\min}}^{\dim X}$ is a positive rational number. In fact the threefold X_{46} given by a degree 46 hypersurface in weighted projective space $\mathbb{P}(4, 5, 6, 7, 23)$, satisfies $K_{X_{\min}}^3 = 1/420$ and ϕ_m is birational if and only if $m = 23$ or $m \geq 27$ (cf. [Iano-Fletcher00]). A further complication is given by the fact that we have little control over other terms of the Riemann-Roch formula for multiples of the canonical bundle (however see Section 2.1 for the 3-fold case). In particular we do not control $\chi(\mathcal{O}_X)$. (This should be contrasted with the above mentioned results for surfaces: $K_{X_{\min}}^2 \geq 1$ and $\chi(\mathcal{O}_X) \geq 1$.)

Using ideas of Tsuji, the following result was proven in [HM06], [Takayama06] and [Tsuji07].

Theorem 1.7. *For any positive integer n, there exists an integer r_n such that if X is a smooth variety of general type and dimension n, then $\phi_r : X \dashrightarrow \mathbb{P}(H^0(X, \omega_X^{\otimes r}))$ is birational for all $r \geq r_n$.*

In fact it turns out that proving the above result is equivalent to showing that the volume

$$\mathrm{vol}(\omega_X) := \lim_{m\to\infty} \frac{n! h^0(\omega_X^{\otimes m})}{m^n}$$

is bounded from below by a positive constant v_n depending only on the dimension $n = \dim X$. We will discuss the ideas behind the proof of this result in Section 2.

Remark 1.8. *Notice that in characteristic $p > 0$ Theorem 1.7 is only known to hold in dimension ≤ 2.*

It should be observed that the proof (1.7) is not effective so that we are unable to compute r_n the minimum integer such that ϕ_r is birational for all n-dimensional varieties of general type and for all $r \geq r_n$.

Recently, effective results were proven for 3-folds of general type. In [Todorov07], it is shown that if $\mathrm{vol}(\omega_X)$ is sufficiently big, then ϕ_m is birational for all $m \geq 5$ (see [DiBiagio10] for related results in dimension 4). In [CC08], J. A. Chen and M. Chen show the following almost optimal result.

Theorem 1.9. *Let X be a smooth projective 3-fold of general type, then ϕ_r is birational for all $r \geq 77$.*

Their proof is based on a detailed analysis of Reid's exact plurigenera formula for threefolds (see also [CC08b], [Zhu09a], [Zhu09b] for related results). In higher dimensions the situation is more complicated and effective results are not known.

Naturally, one may ask whether similar results are known for varieties not of general type. Recall that by definition the Kodaira dimension of a complex projective variety X is given by

$$\kappa(X) = \max\{\dim \phi_m(X) | m \in \mathbb{Z}_{>0}\}.$$

Here we make the convention that if $h^0(\omega_X^{\otimes m}) = 0$ for all $m \in \mathbb{Z}_{>0}$, then $\kappa(X) = -1$ so that $\kappa(X) \in \{-1, 0, 1, \ldots, \dim X\}$. Note that in this case some authors define $\kappa(X) = -\infty$ (instead of $\kappa(X) = -1$) and some others simply say $\kappa(X) < 0$. With our convention we have $\kappa(X) = \mathrm{tr.deg.}_{\mathbb{C}} R(\omega_X) - 1$. (Note that by [BCHM09], the graded ring $R(\omega_X)$ is finitely generated.) Another equivalent definition is $\kappa(X) = \dim \mathrm{Proj}\, R(\omega_X)$. In fact, ϕ_r is birational to the Iitaka fibration and its image is birational to $\mathrm{Proj}\, R(\omega_X)$ for all sufficiently divisible integers $r > 0$. The natural conjecture is then:

Conjecture 1.10. *Fix $n \in \mathbb{Z}_{>0}$ and $\kappa \in \mathbb{Z}_{\geq 0}$. Then there exist a positive integer k_n depending only on n and κ such that for all smooth complex projective varieties of dimension $\dim X = n$ and Kodaira dimension $\kappa(X) = \kappa$, the image of ϕ_r is birational to $\mathrm{Proj}\, R(\omega_X)$ for all integers $r > 0$ divisible by k_n.*

By work of Fujino and Mori cf. [FM00], it is known that there exist positive integers m_1 and m_2 such that

$$R(K_X)^{(m_1)} \cong R(K_Z + B)^{(m_2)}$$

where (Z, B) is a klt pair of general type birational to $\operatorname{Proj} R(\omega_X)$ and for any positive integer m, $R^{(m)} = \oplus_{t \geq 0} R_{mt}$ is the m-th truncation of the graded ring $R = \oplus_{t \geq 0} R_t$. Therefore, this problem is closely related to the natural problem of studying pluricanonical maps for varieties of log general type. These issues will be discussed in Section 3.

Pluricanonical maps for varieties of log general type also arise when studying the automorphism groups of varieties of general type. We now illustrate this in dimension 1.

Theorem 1.11 (Hurwitz). *Let X be a curve of genus $g \geq 2$ with automorphism group G. Then $|G| \leq 84(g-1)$.*

Proof. Let $f : X \to Y = X/G$ be the induced morphism, then

$$K_X = f^* \left(K_Y + \sum \left(1 - \frac{1}{n_i} \right) P_i \right)$$

where n_i is the order of ramification of f over P_i. We have

$$2(g-1) = \deg K_X = |G| \cdot \deg \left(K_Y + \sum \left(1 - \frac{1}{n_i} \right) P_i \right).$$

Therefore, the theorem follows since by (1.12), we have

$$\deg \left(K_Y + \sum \left(1 - \frac{1}{n_i} \right) P_i \right) \geq \frac{1}{42}.$$ □

Theorem 1.12. *Let $\mathcal{A} \subset [0,1]$ be a DCC set (so that any non-increasing sequence $a_i \in \mathcal{A}$ is eventually constant). Then*

$$\mathcal{V} := \{2g - 2 + \sum d_i | g \in \mathbb{Z}_{\geq 0},\ d_i \in \mathcal{A}\} \cap (0,1]$$

is a DCC set and in particular there is a minimal element $v_0 \in \mathcal{V}$.

If $\mathcal{A} = \{1 - \frac{1}{m} | m \in \mathbb{Z}_{>0}\}$, then $v_0 = \frac{1}{42}$.

The proof is elementary, but we recall it for the convenience of the reader.

Proof. We may assume that $g \in \{0,1\}$. It is easy to see that the set $\mathcal{A}_+ = \{\sum a_i | a_i \in \mathcal{A}\} \cap [0,1]$ is also a DCC set and hence so is $\mathcal{V}$.

If $\mathcal{A} = \{1 - \frac{1}{m} | m \in \mathbb{Z}_{>0}\}$ then $v_0 = \sum a_i + 2g - 2$ where $g \in \{0,1\}$ and $a_i = 1 - \frac{1}{n_i}$ for some $n_i \in \mathbb{Z}_{>0}$. If $g = 0$, $a_1 = 1 - \frac{1}{2}$, $a_2 = 1 - \frac{1}{3}$ and $a_3 = 1 - \frac{1}{7}$, then $v_0 = \frac{1}{42}$.

If $g = 1$, then $\sum a_i \geq \frac{1}{2}$. Therefore, we may assume that $g = 0$. In this case $v_0 = \sum_{i=1}^{t} a_i - 2$. Since $1 \geq a_i = 1 - \frac{1}{n_i} \geq \frac{1}{2}$, we may assume $t \in \{3, 4\}$. Let $2 \leq n_1 \leq n_2 \leq \ldots$. If $t = 4$, then as $v_0 > 0$, we have $n_4 \geq 3$ and hence $v_0 = 2 - \sum \frac{1}{n_i} \geq \frac{1}{6}$. If $t = 3$, then $v_0 = 1 - \sum \frac{1}{n_i}$. If $n_1 > 3$, then $v_0 \geq \frac{1}{4}$. If $n_1 = 3$, as $v_0 > 0$, we have $n_3 \geq 4$ and hence $v_0 \geq \frac{1}{12}$. If $n_1 = 2$ and $n_2 \geq 5$, then $v_0 \geq \frac{1}{10}$. If $n_1 = 2$ and $n_2 = 4$, then as $v_0 > 0$, $n_3 \geq 5$ and so $v_0 \geq \frac{1}{20}$. If $n_1 = 2$ and $n_2 = 3$, then as $v_0 > 0$, $n_3 \geq 7$ and so $v_0 \geq \frac{1}{42}$. Finally, if $n_1 = n_2 = 2$, then $v_0 < 0$. □

One expects results similar to (1.11) to hold for automorphism groups of varieties of general type (regardless of their dimension). Results in this direction will be discussed in Section 3.1.

Another reason to be interested in pluricanonical maps for varieties of log general type is that they naturally arise when studying moduli spaces of canonically polarized varieties of general type cf. Section 3.3 and open varieties cf. Section 3.4.

At the opposite end of the spectrum, we have varieties with $\kappa(X) < 0$. ¿From the point of view of the minimal model program, the typical representatives of this class of varieties are Fano varieties. For these varieties we have that $\omega_X^\vee$ is ample. Therefore, we consider the maps induced by sections of $\omega_X^{\otimes m}$ for $m \in \mathbb{Z}_{<0}$. The geometry of Fano varieties is briefly discussed in Section 3.5.

2. Varieties of General Type

In this section we will explain the main ideas behind the proof of (1.7). Our goal is to show that if X is an n-dimensional projective variety of general type, then ϕ_r is birational for all $r \gg r_n$. To this end, it suffices then to show that there exists a subset $X^0 \subset X$ given by the complement of countably many closed subsets of X such that ϕ_r is defined at points of X^0 and ϕ_r separates any two distinct points $x, y \in X^0$. The first major reduction in the proof of (1.7) is to show the following.

Proposition 2.1. *In order to prove* (1.7) *it suffices to show that there exist positive constants A and B (depending only on n) such that for any integer*

$$r \geq \frac{A}{\mathrm{vol}(\omega_X)^{1/n}} + B,$$

the rational map ϕ_r is birational.

Proof. If $\mathrm{vol}(\omega_X) \geq 1$, then the assertion is clear as ϕ_r is birational for all $r \geq A + B$. We may therefore assume that $\mathrm{vol}(\omega_X) < 1$. Let r_0 be the smallest integer such that ϕ_{r_0} is birational, then

$$1 \leq \deg \overline{\phi_{r_0}(X)} \leq \mathrm{vol}(\omega_X^{\otimes r_0}) = r_0^n \mathrm{vol}(\omega_X) \leq$$

$$\left(\frac{A}{\mathrm{vol}(\omega_X)^{1/n}} + B + 1\right)^n \mathrm{vol}(\omega_X) < (A+B+1)^n.$$

It follows that the degree of the closure of $\phi_{r_0}(X)$ is bounded. Therefore, by a Hilbert scheme type argument, there is a projective morphism of quasi-projective varieties $f : \mathcal{X} \to S$ such that if X is any smooth n-dimensional complex projective variety with $0 < \mathrm{vol}(\omega_X) < 1$, then there exists a point $s \in S$ such that X is birational to the fiber $\mathcal{X}_s$. By Noetherian induction, possibly replacing S by a union of locally closed subsets, we may assume that f is smooth and S is irreducible. Let $\eta = \mathrm{Spec}(K)$ be the generic point of S and $\mathcal{X}_K$ be the generic fiber. Then there exists r_η such that $\phi_{\omega^{\otimes r}_{\mathcal{X}_K}}$ is birational for all $r_\eta \leq r \leq 2r_\eta$ (and hence for all $r \geq r_\eta$). It then follows that there exists an open subset S^0 of S such that $\phi_{\omega^{\otimes r}_{\mathcal{X}_t}}$ is birational for all $t \in S^0$ and all $r_\eta \leq r \leq 2r_\eta$ (and hence for all $r \geq r_\eta$).

By Noetherian induction, there is an integer r_S such that $\phi_{\omega^{\otimes r}_{\mathcal{X}_t}}$ is birational for all $t \in S$ and all $r \geq r_S$. $\square$

Remark 2.2. *By the above discussion,* (1.7) *implies that for any* $n \in \mathbb{Z}_{>0}$, *there exist a positive constant* $v_n > 0$ *such that if* X *is a projective variety of general type and* $\dim X = n$, *then* $\mathrm{vol}(\omega_X) \geq v_n$.

In order to show that a rational map ϕ_r is birational, we would like to imitate the proof of the curve case of this theorem cf. (1.1) and show that the evaluation map

$$H^0(X, \omega_X^{\otimes r}) \to \mathbb{C}_x \oplus \mathbb{C}_y$$

at very general points $x, y \in X$ is surjective. The problem is that in higher dimensions it is very hard to ensure that cohomology groups of the form $H^1(X, \omega_X^{\otimes r} \otimes m_x \otimes m_y)$ vanish (here m_x denotes the maximal ideal of $x \in X$). In order to achieve this, the usual strategy is to use a far reaching generalization of Kodaira vanishing known as Kawamata-Viehweg vanishing or Nadel vanishing. Recall the following:

Theorem 2.3 (Nadel vanishing). *Let* X *be a smooth complex projective variety,* $\mathcal{L}$ *a line bundle on* X *and* D *a* $\mathbb{Q}$*-divisor such that* $\mathcal{L}(-D)$ *is nef and big. Then* $H^i(X, \omega_X \otimes \mathcal{L} \otimes \mathcal{J}(D)) = 0$ *for all* $i > 0$.

Remark 2.4. *Recall that a line bundle is nef if* $\deg(\mathcal{L}|_C) \geq 0$ *for any curve* $C \subset X$. *In this case,* $\mathcal{L}$ *is big if and only if* $\mathcal{L}^{\dim X} > 0$. *These definitions readily extend to* $\mathbb{Q}$*-divisors.*

Remark 2.5. *Recall that if* $D \subset X$ *is a* $\mathbb{Q}$*-divisor, then the multiplier ideal* $\mathcal{J}(D) \subset \mathcal{O}_X$ *is defined as follows. Let* $f : Y \to X$ *be a log resolution so that* f

is a projective birational morphism, Y is smooth, $\operatorname{Exc}(f)$ *and* $\operatorname{Exc}(f) \cup f_*^{-1}D$ *are divisors with simple normal crossings support. Then*

$$\mathcal{J}(D) := f_*\mathcal{O}_Y(K_{Y/X} - \llcorner f^*D \lrcorner).$$

It is well known that $\mathcal{J}(D)$ is trivial at points $x \in X$ where $\operatorname{mult}_x(D) < 1$ *and $m_x \subset \mathcal{J}(D)$ if* $\operatorname{mult}_x(D) \geq \dim X$. *The interested reader can consult [Lazarsfeld05] for a clear and comprehensive treatment of the properties of multiplier ideal sheaves.*

Using Nadel vanishing we obtain the following.

Proposition 2.6. *In order to prove* (1.7) *it suffices to show that there exists positive constants A and B (depending only on n) such that for any two distinct very general points $x, y \in X$ there is a $\mathbb{Q}$-divisor $D_{x,y}$ such that*

1. *$D_{x,y} \sim \lambda K_X$ where $\lambda < \frac{A}{\operatorname{vol}(\omega_X)^{1/n}} + B - 1$;*
2. *x is an isolated point of the co-support of $\mathcal{J}(D_{x,y})$ and y is contained in the co-support of $\mathcal{J}(D_{x,y})$.*

Proof. Let $r \geq \frac{A}{\operatorname{vol}(\omega_X)^{1/n}} + B$ be any integer. By (2.1), it suffices to show that ϕ_r is birational.

Since ω_X is big, there exists an integer $m > 0$, an ample divisor H and an effective divisor $G \geq 0$ such that $mK_X \sim G + H$. We may assume that x, y are not contained in the support of G. We let $D'_{x,y} = D_{x,y} + \frac{r-1-\lambda}{m}G$. Then $(r-1)K_X - D'_{x,y} \sim_{\mathbb{Q}} \frac{r-1-\lambda}{m}H$ is ample so that by (2.3) $H^1(X, \omega_X^{\otimes r} \otimes \mathcal{J}(D'_{x,y})) = 0$.

Consider the short exact sequence of coherent sheaves on X

$$0 \to \omega_X^{\otimes r} \otimes \mathcal{J}(D'_{x,y}) \to \omega_X^{\otimes r} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ denotes the corresponding quotient. Since, as observed above, $H^1(X, \omega_X^{\otimes r} \otimes \mathcal{J}(D'_{x,y})) = 0$, the homomorphism

$$H^0(X, \omega_X^{\otimes r}) \to H^0(X, \mathcal{Q})$$

is surjective. Since x is an isolated point in the co-support of $\mathcal{J}(D'_{x,y})$, $\mathbb{C}_x$ is a summand of $\mathcal{Q}$. Since y is also contained in the support of $\mathcal{Q}$, we may find a section $s \in H^0(X, \omega_X^{\otimes r})$ vanishing at y but not at x. Since x and y are very general points on X, by symmetry we may also find a section $t \in H^0(X, \omega_X^{\otimes r})$ vanishing at x but not at y. It follows that the evaluation map

$$H^0(X, \omega_X^{\otimes r}) \to \mathbb{C}_x \oplus \mathbb{C}_y$$

is surjective and hence ϕ_r is birational. $\square$

Proof of Theorem 1.7. By (2.6), it suffices to show that there exists positive constants A and B (depending only on n) such that for any two distinct very general points $x, y \in X$ there is a $\mathbb{Q}$-divisor $D_{x,y} \sim_{\mathbb{Q}} \lambda K_X$ where $\lambda < \frac{A}{\operatorname{vol}(\omega_X)^{1/n}} + B - 1$ such that x is an isolated point of the co-support of $\mathcal{J}(D_{x,y})$ and y is contained in the co-support of $\mathcal{J}(D_{x,y})$.

For ease of exposition, we will however just show that there is a $\mathbb{Q}$-divisor $D_x \sim_{\mathbb{Q}} \lambda K_X$ where $\lambda < \frac{A}{\operatorname{vol}(\omega_X)^{1/n}} + B - 1$ such that x is an isolated point of the co-support of $\mathcal{J}(D_x)$. The interested reader can consult [Tsuji07] or [Takayama06] for the remaining details or [HM06] for an alternative argument.

We will also assume that ω_X is ample. This can be achieved replacing X by its canonical model. Of course X is no longer smooth, but it has mild (canonical) singularities and the proof goes through with minor changes.

We will proceed by induction on the dimension and hence we may assume that (1.7) holds for varieties of dimension $\leq n-1$. Note that by (1.1), the theorem holds when $n = 1$. We will not keep careful track of the various constants and so we will say that $\lambda = O(\operatorname{vol}(\omega_X)^{-1/n})$ (instead of $\lambda < \frac{A}{\operatorname{vol}(\omega_X)^{1/n}} + B - 1$).

Since

$$h^0(\mathcal{O}_X(mK_X)) = \frac{\operatorname{vol}(\omega_X)}{n!} m^n + O(m^{n-1})$$

and since vanishing to order k at a smooth point $x \in X$ imposes at most $k^n/n! + O(k^{n-1})$ conditions, by an easy calculation it follows that for any smooth point $x \in X$, we may find $m \gg 0$ and a $\mathbb{Q}$-divisor $D_x^m \sim mK_X$ such that $\operatorname{mult}_x(D_x^m) > \frac{m}{2}\operatorname{vol}(\omega_X)^{1/n}$. Note that if we assume that $x \in X$ is a very general point, then we can assume that the integer m is independent of the point x. Let τ be defined by

$$\tau = \sup\{t \geq 0 | m_x \subset \mathcal{J}(X, tD_x^m)\}.$$

By (2.5), $\tau < \frac{2n}{m \cdot \operatorname{vol}(\omega_X)^{1/n}}$. Note that if $D_x := \tau D_x^m$, then $m_x \subset \mathcal{J}(X, D_x)$ and $D_x \sim \lambda K_X$ where $\lambda \leq \frac{2n}{\operatorname{vol}(\omega_X)^{1/n}}$ so that $\lambda = O(\operatorname{vol}(\omega_X)^{-1/n})$.

By a standard perturbation technique, we may assume that on a neighborhood of $x \in X$ there is a unique irreducible subvariety V_x contained in the co-support of $\mathcal{J}(D_x)$. (More precisely, if $f : Y \to X$ is a log resolution of (X, D_x), we may assume that there is a unique divisor $E \subset Y$ such that $\operatorname{mult}_E(K_{Y/X} - f^*D_x) = -1$ and $E \cap f^{-1}(x) \neq \emptyset$. V_x is then the center of E on X.) The problem is that we may have $\dim V_x > 0$. The idea is to then use the techniques of [AS95] to "cut down" the cosupport of $\mathcal{J}(D_x)$ i.e. to reduce to the case when $\dim V_x = 0$. We will use the following result:

Proposition 2.7. *Let V_x and (X, D_x) be as above. If for a general point $x' \in V_x$ there exists a divisor $F_{x'}$ on X whose support does not contain V_x such that $\operatorname{mult}_{x'}(F_{x'}|_{V_x}) > \dim V_x$, then there exist rational numbers $0 < \alpha, \beta < 1$ such that $m_{x'} \subset \mathcal{J}(\alpha D_x + \beta F_{x'})$ and in a neighborhood of x', every component of the co-support of $\mathcal{J}(\alpha D_x + \beta F_{x'})$ has dimension less than $\dim V_x$.*

The established strategy to produce the $\mathbb{Q}$-divisor $F_{x'}$ is as follows:

1. produce a divisor $E_{x'}$ on V_x such that $\mathrm{mult}_{x'}(E_{x'}) > \dim V_x$, and then
2. lift this divisor to X, that is find a $\mathbb{Q}$-divisor $F_{x'} \sim_{\mathbb{Q}} \lambda' K_X$ such that $F_{x'}|_{V_x} = E_{x'}$ and $\lambda' = O(\mathrm{vol}(\omega_X)^{-1/n})$.

In order to complete the first step, we need to bound the volume of $\omega_X|_{V_x}$ from below. This is achieved by comparing $K_X + D_x$ with K_{V_x} via a result of Kawamata (cf. [Kawamata98]):

Theorem 2.8. *Let V_x and (X, D_x) be as above, and let A be an ample divisor. If $\nu : V_x^\nu \to V_x$ is the normalization, then for any rational number $\epsilon > 0$, there exists a $\mathbb{Q}$-divisor $\Delta_\epsilon \geq 0$ such that*

$$\nu^*(K_X + D_x + \epsilon A) \sim_{\mathbb{Q}} K_{V_x^\nu} + \Delta_\epsilon.$$

Remark 2.9. *Kawamata's Subadjunction Theorem says that if moreover V_x is a minimal non-klt center at a point $y \in V_x$, then (on a neighborhood of y) V is normal and we may assume that (V_x, Δ_ϵ) is klt.*

Since X is of general type and $x \in X$ is a very general point, it follows that V_x is also of general type. Let $n' = \dim V_x$ and $\mu : \tilde{V}_x \to V_x^\nu$ be a resolution of singularities. Assume for simplicity that V_x is normal. By our inductive hypothesis, for general $x' \in \tilde{V}_x$ there is a $\mathbb{Q}$-divisor $E_{x'} \sim_{\mathbb{Q}} \gamma K_{\tilde{V}_x}$ on $\tilde{V}_x$ with $\mathrm{mult}_{x'}(E_{x'}) > n'$ and $0 < \gamma < n/v_{n'}$ so that $\gamma = O(1)$ (for the definition of $v_{n'}$ see (2.2)). Fix a rational number $0 < \epsilon \ll 1$ and let $A = K_X$. Pushing forward, we obtain a $\mathbb{Q}$-divisor

$$\nu_*(\mu_* E_{x'} + \gamma \Delta_\epsilon) \sim_{\mathbb{Q}} \gamma \nu_*(K_{V_x^\nu} + \Delta_\epsilon) \sim_{\mathbb{Q}} \gamma(1 + \lambda + \epsilon) K_X|_{V_x}$$

on V_x with $\mathrm{mult}_{x'} \nu_*(\mu_* E_{x'} + \gamma \Delta_\epsilon) > n'$.

Since we have assumed that K_X is ample, by Serre vanishing, the homomorphism

$$H^0(X, \mathcal{O}_X(mK_X)) \to H^0(X, \mathcal{O}_{V_x}(mK_X))$$

is surjective for all $m \gg 0$ and so there exists a $\mathbb{Q}$-divisor $F_{x'} \sim_{\mathbb{Q}} \gamma(1+\lambda+\epsilon)K_X$ such that $F_{x'}|_{V_x} = \nu_*(\mu_* E_{x'} + \gamma \Delta_\epsilon)$.

By (2.7), we then have that for some $0 < \alpha, \beta < 1$

1. $m_{x'} \subset \mathcal{J}(\alpha D_x + \beta F_{x'})$,
2. in a neighborhood of x', every component of the co-support of $\mathcal{J}(\alpha D_x + \beta F_{x'})$ has dimension $< \dim V_x$, and
3. $\alpha D_x + \beta F_{x'} \sim_{\mathbb{Q}} \lambda' K_X$ where $\lambda' = O(\mathrm{vol}(\omega_X)^{-1/n})$.

Repeating this procedure at most $n-1$ times, we may assume that for any very general point $x^* \in X$, there is a $\mathbb{Q}$-divisor $D^*_{x^*} \sim_{\mathbb{Q}} \lambda^* K_X$ such that x^* is an isolated point in the co-support of $\mathcal{J}(D^*_{x^*})$ and $\lambda^* = O(\mathrm{vol}(\omega_X)^{-1/n})$. $\square$

2.1. Reid's 3-fold exact plurigenera formula. In dimension 3, an almost optimal version of (1.7) can be obtained using Reid's 3-fold exact plurigenera formula.

Theorem 2.10. *Let X be a minimal 3-fold with terminal singularities, then*

$$\chi(\mathcal{O}_X(mK_X)) = \frac{1}{12}m(m-1)(2m-1)K_X^3 - (2m-1)\chi(\mathcal{O}_X) + l(m),$$

where the correction term $l(m)$ depends only on the (finitely many) singularities of X. More precisely, there is a finite set (basket) of pairs of integers $\mathcal{B}(X) = \{(b_i, r_i)\}$ where $0 < b_i < r_i$ are uniquely determined by the singularities of X such that

$$l(m) := \sum_{Q_i \in \mathcal{B}(X)} l_{Q_i}(m) := \sum_{Q_i \in \mathcal{B}(X)} \sum_{j=1}^{m-1} \frac{\overline{jb_i}(r_i - \overline{jb_i})}{2r_i},$$

where $\overline{x}$ denotes the smallest non-negative residue modulo r_i, so that, $\overline{x} := x - r_i \lfloor \frac{x}{r_i} \rfloor$.

When X is of general type, K_X is nef and big so that by Kawamata-Viehweg vanishing we have

$$P_m(X) := h^0(\mathcal{O}_X(mK_X)) = \chi(\mathcal{O}_X(mK_X)) \qquad \text{for all } m \geq 2.$$

One can therefore hope to use (2.10) to find values of m such that $P_m(X) \geq 1$ or $P_m(X) \geq 2$. If, for example $\chi(\mathcal{O}_X) \leq 0$, then since $l(m) \geq 0$ and $K_X^3 > 0$, we have $P_m(X) \geq 1$ for all $m \geq 2$.

More generally, it is not hard to see that if $P_m(X) = 0$ for some $m \geq 2$ and if $-\chi(\mathcal{O}_X)$ is bounded from below, then there are only finitely many possible baskets of singularities $\mathcal{B}(X)$. This implies that the index r of K_X (i.e. the smallest integer $r > 0$ such that rK_X is Cartier) is bounded from above. In turn this means that $K_X^3 \geq \frac{1}{r^3}$ and hence one obtains an integer m_0 such that $P_m(X) \geq 1$ for all $m \geq m_0$.

By a detailed study of the above Riemann-Roch formula, J.-A. Chen and M. Chen prove the following (cf. [CC08]).

Theorem 2.11. *Let X be a non-singular 3-fold of general type then*

1. $\mathrm{vol}(\omega_X) \geq \frac{1}{2660}$,
2. $P_{12}(X) \geq 1$,
3. $P_{24}(X) \geq 2$, *and*
4. ϕ_r *is birational for all* $r \geq 77$.

Remark 2.12. *The second and third inequalities are optimal.*

There exist examples with $\mathrm{vol}(\omega_X) = \frac{1}{420}$ *(cf. [Iano-Fletcher00]) and hence the first inequality is "almost optimal". By [CC08b], it is known that if* $\chi(\mathcal{O}_X) \leq 0$*, then* $\mathrm{vol}(\omega_X) \geq \frac{1}{30}$*. This inequality is optimal as shown by the example of a canonical hypersurface of degree* 28 *in the weighted projective space* $\mathbb{P}(1, 3, 4, 5, 14)$*.*

When $\chi(\mathcal{O}_X) = 1$*, it is known that* $\mathrm{vol}(\omega_X) \geq \frac{1}{420}$ *(cf. [Zhu09b]) and that* ϕ_r *is birational for all* $r \geq 46$ *(cf. [Zhu09a]).*

As mentioned in the introduction, there are examples where ϕ_{26} *is not birational and so the fourth inequality is also "almost optimal".*

Remark 2.13. *Using similar methods, in [CC08c] it is shown that if* X *is a terminal weak* $\mathbb{Q}$*-Fano* 3*-fold (so that* $-K_X$ *is nef and big), then* $h^0(\mathcal{O}_X(-6K_X)) > 0$*,* $h^0(\mathcal{O}_X(-8K_X)) > 1$ *and* $-K_X^3 \geq \frac{1}{330}$ *(which is the optimal possible lower bound).*

As mentioned above, the idea of using Reid's exact plurigenera formula in this context is not new (see for example [Iano-Fletcher00]). The main new insight of [CC08] is to use (2.10) for various values of m to prove the following inequality:

$$2P_5 + 3P_6 + P_8 + P_{10} + P_{12} \geq \chi(\mathcal{O}_X) + 10P_2 + 4P_3 + P_7 + P_{11} + P_{13}.$$

It follows that if $P_m = 0$ for $m \leq 12$, then $\chi(\mathcal{O}_X) \leq 0$ which as observed above is the well understood case.

The precise results obtained in [CC08] are then a consequence of a detailed study of the terms appearing in Reid's exact plurigenera formula.

3. Varieties of Log General Type

One would like to generalize Theorem 1.7 to the case of log canonical pairs. This is a natural question in its own right, but it is also motivated by the desire to study the geometry of open varieties, of varieties of intermediate Kodaira dimension, of the moduli spaces of varieties of general type, of the automorphism groups of varieties of general type and other related questions.

We start by considering the case of curves. Let (X, D) be a pair consisting of a smooth curve X and a $\mathbb{Q}$-divisor $D = \sum d_i D_i$ such that $K_X + D$ has general type. We ask the following:

Question 3.1. *Is there a lower bound for the volume of* $K_X + D$*?*

The answer in this case is simple:

$$\mathrm{vol}(K_X + D) = \deg(K_X + D) = 2g - 2 + \sum d_i > 0$$

where g denotes the genus of the curve X. If $g \geq 2$, then $\mathrm{vol}(K_X + D) \geq 2$, but if $g \leq 1$, one sees immediately that no such bound exists unless we impose some restrictions on the possible values that d_i are allowed to take. The most natural answer was given in (1.12): *If* $\mathcal{A} \subset [0,1]$ *is a DCC set, then there exists a constant* $v_0 > 0$ *such that* $2g - 2 + \sum d_i \geq v_0$ *for any* $g \in \mathbb{Z}_{\geq 0}$ *and* $d_i \in \mathcal{A}$*.*

The most optimistic generalizations of Theorem 1.12 are the following two conjectures.

Conjecture 3.2. *Let* $\mathcal{A} \subset (0,1]$ *be a DCC set,* $n \in \mathbb{Z}_{>0}$ *and*

$$\mathcal{V} = \{\mathrm{vol}(K_X + D) | (X, D) \text{ is lc, } \dim X = n, \ D \in \mathcal{A}\}.$$

Then $\mathcal{V}$ *is a DCC set.*

Conjecture 3.3. *Let* $\mathcal{A} \subset (0,1]$ *be a DCC set and* $n \in \mathbb{Z}_{>0}$*. Then there exists a positive integer* $N > 0$ *such that if* (X, D) *is a lc pair of dimension* n *with* $K_X + D$ *big and* $D \in \mathcal{A}$*, then* $|\llcorner m(K_X + D) \lrcorner|$ *is birational for all* $m \geq N$*.*

Notice that the above conjectures were proven in dimension 2 by Alexeev and Alexeev-Mori (cf. [Alexeev94] and [AM04]).

At first sight one may hope to apply the techniques used in the proof of Theorem 1.7, however there are several problems that arise:

It is easy to produce a divisor $D_x \sim_{\mathbb{Q}} k(K_X + D)$ such that $m_x \subset \mathcal{J}(D_x)$ for very general $x \in X$ and $k = O(\mathrm{vol}(K_X + D)^{1/n})$. Assume for simplicity that there is an irreducible subvariety $V_x \subset X$ such that $\mathcal{J}(D_x) = \mathcal{I}_{V_x}$ on a neighborhood of $x \in X$. If $\dim V_x > 0$, we must bound $\mathrm{vol}((K_X + D)|_{V_x})$ from below. To this end, one applies Kawamata sub-adjunction

$$\nu^*(K_X + D_x + \epsilon A) = K_{V_x^\nu} + \Delta_\epsilon$$

where $\nu : V_x^\nu \to V_x$ is the normalization morphism, A is an ample line bundle and $0 < \epsilon \ll 1$.

In order to proceed by induction on the dimension, we must show that $K_{V_x^\nu} + \Delta_\epsilon$ satisfies the inductive hypothesis. This is problematic. Even if we ignore the dependence on ϵ (which is at least conjecturally a reasonable assumption), in order to control the coefficients of Δ_ϵ, we must control the coefficients of D_x. In higher dimension, there is no known strategy to accomplish this.

3.1. Automorphism groups of varieties of general type.

Let X be a variety of general type with automorphism group G, then it is known that G is finite. It is a natural question to find effective bounds on the order of G.

Naturally, one would hope to generalize Hurwitz's Theorem cf. (1.11) to higher dimensions. The natural conjecture is:

Conjecture 3.4. *For any* $n \in \mathbb{Z}_{>0}$*, there exists a constant* $C > 0$ *(depending only on* n*) such that if* X *is an* n*-dimensional variety of general type with automorphism group* G*, then*

$$|G| \leq C \cdot \mathrm{vol}(\omega_X).$$

Over the years there has been much interest in results related to the above conjecture; see for example [Andreotti50], [Corti91], [HS91], [Xiao94], [Xiao95], [Xiao96], [Szabo96], [CS96], [Ballico93] and [Cai00].

One would hope to use the ideas in the proof of (1.11) to attack Conjecture 3.4 in higher dimensions. We can still write

$$K_X = f^* \left(K_Y + \sum (1 - \frac{1}{n_i}) P_i \right)$$

where $f : X \to Y = X/G$ is the induced morphism and n_i is the order of ramification of f over P_i. We also have

$$\mathrm{vol}(\omega_X) = |G| \cdot \mathrm{vol} \left(K_Y + \sum (1 - \frac{1}{n_i}) P_i \right).$$

Therefore, a positive answer to Conjecture 3.2, would imply a positive answer to Conjecture 3.4.

Remark 3.5. *It is likely that proving that* $\mathrm{vol}(K_Y + \sum(1 - \frac{1}{n_i})P_i)$ *is bounded from below, is substantially easier than Conjecture 3.2, and that this problem is even more accessible when* $(Y, \sum(1 - \frac{1}{n_i})P_i)$ *arises as the quotient of a variety of general type by its automorphism group.*

3.2. Varieties of intermediate Kodaira dimension.

Let X be a smooth projective variety of Kodaira dimension $0 \leq \kappa(X) < \dim X$, then it is known that for all $m > 0$ sufficiently divisible $\phi_m : X \to Z$ defines a map birational to the Iitaka fibration so that $\dim Z = \kappa(X)$ and $\kappa(F) = 0$ where F is a general fiber of ϕ_m. (In fact Z is birational to $\mathrm{Proj} R(K_X)$.) When $\kappa(X) = 0$, $Z = \mathrm{Spec}(k)$ and there is an integer $N > 0$ such that $P_m(X) > 0$ if and only if m is divisible by N.

It is natural to conjecture the following:

Conjecture 3.6. *Fix positive integers* $0 \leq \kappa < n$. *Then there exists an integer* $N > 0$ *(depending only on* κ *and* n*) such that if* X *is a smooth projective variety of dimension* n *and Kodaira dimension* κ*, and* $m > 0$ *is an integer divisible by* N*, then* ϕ_N *is birational to the Iitaka fibration.*

For surfaces, this conjecture is known to be true. In fact we have the following:

1. If $\kappa(X) = 0$ then $P_{12}(X) > 0$, and
2. if $\kappa(X) = 1$, then $P_{12}(X) > 0$ and $P_m(X) > 1$ for some $m \leq 42$.

In dimension 3 the following results are known:

1. If $\kappa(X) = 0$ then by [Kawamata86] and [Morrison86]

$$P_{2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19}(X) > 0;$$

2. if $\kappa(X) = 1$, then by [FM00] there exists an explicit constant $N > 0$ (presumably far from optimal) such that ϕ_m is birational to the Iitaka fibration for all $m > 0$ divisible by N; and

3. if $\kappa(X) = 2$, then by [VZ09] and [Ringler07], there exists an explicit constant $N > 0$ such that ϕ_m is birational to the Iitaka fibration for all $m > 0$ divisible by N (in fact $m \geq 48$ and divisible by 12 suffices).

We now outline a strategy for proving Conjecture (3.6). Let X be a smooth projective variety of dimension n and Kodaira dimension $\kappa \geq 0$.

Step 1. By the minimal model program, it is expected that there is a minimal model $\phi : X \dashrightarrow X'$ (given by a finite sequence of flips and divisorial contractions) such that $R(K_X) \cong R(K_{X'})$, X' has terminal singularities and $K_{X'}$ is semiample. This means that for some $m_0 > 0$, the linear series $|m_0 K_{X'}|$ is base point free and it defines a morphism $f' : X' \to Z'$ which is birational to the Iitaka fibration of X. In particular $\dim Z = \kappa$ and $\kappa(F') = 0$ where F' is a very general fiber of f'. In fact we have $K_{F'} \sim_{\mathbb{Q}} 0$. Note this step requires the abundance conjecture.

Step 2. Using the ideas of Fujino and Mori [FM00], we write the "canonical bundle formula"

$$K_{X'} \sim_{\mathbb{Q}} {f'}^*(K_Z + B + M)$$

where the "boundary" part B is determined by the singularities of the morphism f' and the "moduli" part M is determined by the variation in moduli of the general fiber F'.

When f' is an elliptic fibration, then $M = \frac{1}{12} j^*\mathcal{O}_{\mathbb{P}^1}(1)$ where $j : Z \to \mathbb{P}^1$ is the j-function. In general, one expects M to be the pull-back of a big semiample $\mathbb{Q}$-divisor on a moduli scheme.

In order to make use of Fujino-Mori's canonical bundle formula, it is important to bound the denominators of the $\mathbb{Q}$-divisors B and M.

By [FM00, 3.1], there exists a positive integer $k = k(b, B_m) > 0$ such that kM is a divisor, where b is the smallest positive integer such that $P_b(F') > 0$, $m = n - \kappa$ and B_m is the m-th Betti number of a desingularization of the $\mathbb{Z}_m$-cover $E \to F'$ determined by the divisor in $|bK_{F'}|$. In fact we have $k = \operatorname{lcm}\{y \in \mathbb{Z}_{>0} | \phi(y) \leq B_m\}$ where ϕ is Euler's function.

The boundary part, B is defined as follows: Let P be a codimension 1 point on Z and let b_P be the supremum of $b \geq 0$ such that (X, bf^*P) is log canonical over the general point of P. We then set $B = \sum(1 - b_P)P$. Note that $b_P = 1$ for all but finitely many codimension 1 points P on Z. An interesting feature is that the coefficients of b_p are of the form $b - \frac{v}{ku}$ where $0 < v \leq bk$ cf. [FM00, 2.8].

The upshot is that if we control the invariants b and B_m of the general fiber F', then we can bound the denominators of B and M.

Step 3. Apply Conjecture (3.3) to conclude.

Remark 3.7. *Note that Step 2 depends on bounding k independently of the general fiber F'. If $m = 1$ then F' is a curve of genus 1 and hence $b = 1$ and $B_1 = 2$. If $m = 2$, then F is either an abelian, a K3, and Enriques or a bielliptic surface. As mentioned above, we have $b \leq 12$. Let $E \to F'$ be the corresponding cover. We again have $\kappa(E) = 0$ and hence $B_2(E) \leq 24$. If $m = 3$, by Kawamata's result mentioned above, there is a known bound for b, but there is no known bound for B_3. In higher dimensions, these questions are completely open.*

Remark 3.8. *One may make the analogous conjecture for log pairs. The case when $\dim X \leq 3$ and $\kappa(K_X + \Delta) = \dim X - 1$ is treated in [Todorov08]. The case where $\dim X \leq 4$ and $\kappa(K_X + \Delta) = \dim X - 2$ is treated in [TX08].*

3.3. Moduli spaces of varieties of general type.

As we have remarked above, boundedness of varieties of general type is an essential ingredient in the proof of the existence of a moduli space for canonically polarized varieties of general type.

Recall that if X is a projective variety of general type, then its canonical model X_{can} is defined by $X_{\text{can}} := \text{Proj}R(K_X)$. X_{can} has canonical singularities (in particular $K_{X_{\text{can}}}$ is $\mathbb{Q}$-Cartier and $R(K_{X_{\text{can}}}) \cong R(K_X)$) and $K_{X_{\text{can}}}$ is ample. As a consequence of (1.7), we have:

Theorem 3.9. *For every $n, v \in \mathbb{Z}_{>0}$ then there exists a projective morphism of normal quasi-projective varieties $f : \mathcal{X} \to B$ such that any fiber $\mathcal{X}_b$ is a canonically polarized variety of general type with canonical singularities, and if X is a canonically polarized variety of general type with canonical singularities and $\text{vol}(\omega_X) \leq v$, then there exists $b \in B$ such that $X \cong \mathcal{X}_b$.*

Idea of the proof. By (1.7) and its proof, there exists a projective morphism of normal quasi-projective varieties $f : \mathcal{Z} \to B$ such that for any X as above, there exists $b \in B$ such that $\mathcal{Z}_b$ is birational to X. Note that

$$X \cong \text{Proj}R(K_X) \cong \text{Proj}R(K_{Y_b})$$

where $Y_b \to \mathcal{Z}_b$ is any log resolution.

We may assume that B is irreducible. Let $\eta = \text{Spec}(K)$ be its generic point and $\mathcal{Y}_\eta \to \mathcal{Z}_\eta$ be a log resolution. By [BCHM09], it follows that $R(K_{\mathcal{Y}_\eta})$ is finitely generated. We may therefore pick an integer $N > 0$ such that $R(NK_{\mathcal{Y}_\eta})$ is generated in degree 1. There is an open subset $B^0 \subset B$ such that $R(NK_{\mathcal{Y}^0})$ is generated over B^0 in degree 1 where $\mathcal{Y}^0 = \mathcal{Y} \times_B B^0$. By Noetherian induction, we may assume that $R(NK_{\mathcal{Y}})$ is generated over B in degree 1. Replacing $\mathcal{Y}$ by an appropriate resolution, we may assume that $|NK_{\mathcal{Y}}|$ defines a morphism $\phi_N : \mathcal{Y} \to \mathcal{X} \cong \text{Proj}_B R(K_{\mathcal{Y}})$. We then have

$$X \cong \text{Proj}R(K_{\mathcal{Y}_b}) \cong \text{Proj}R(K_X).$$

□

Ideally, one would like to construct **proper** moduli spaces for varieties of general type. In order to do this, it is necessary to allow certain degenerations of these varieties. For example in dimension 1 it is necessary to consider stable curves and in higher dimensions we must consider semi-log canonical varieties i.e. varieties X such that

1. X is reduced and S_2,
2. K_X is $\mathbb{Q}$-Cartier, and
3. if $f : \tilde{X} \to X$ is a semiresolution of singularities, then $K_{\tilde{X}} \equiv f^*K_X + \sum a_i E_i$ where $a_i \geq -1$.

This is the generalization of log canonical singularities to the non-normal situation.

If we let $\nu : X^\nu \to X$ be the normalization, then $X^\nu = \coprod_{i=1,\ldots,m} X_i$ and we may write $K_{X_i} + \Delta_i = (\nu^* K_X)|_{X_i}$ where (X_i, Δ_i) is a log canonical pair and Δ_i is a reduced divisor.

If we are to construct proper moduli spaces, it is therefore important to prove the boundedness of n-dimensional canonically polarized semi log canonical varieties X with fixed volume $K_X^n = M$.

The first step is provided by an affirmative answer to Conjecture 3.2: Since $K_X^n = \sum_{i=1,\ldots,m} (K_{X_i} + \Delta_i)^n$ and since by (3.2) the numbers $(K_{X_i} + \Delta_i)^n$ belong to a DCC set $\mathcal{V}$, then there exists a positive constant $v > 0$ such that $(K_{X_i} + \Delta_i)^n \geq v$ for all i. In particular there is an upper bound for the number of irreducible components of X i.e. $m \leq M/v$. Moreover, by (3.3) and arguing as in the proof of (3.9), one expects that the pairs (X_i, Δ_i) (and hence the variety X) belong to a bounded family.

3.4. Open varieties. Let X be a smooth quasi-projective variety, and consider $\bar{X}$ a smooth projective variety such that $X = \bar{X} - F$ where F is a simple normal crossing divisor on $\bar{X}$.

The geometry of X is then studied in terms of the rational maps defined by $H^0(\omega_{\bar{X}}^{\otimes m}(mF))$ for $m > 0$. Note these maps are independent of the chosen compactification $\bar{X}$ of X. Conjectures 3.2 and 3.3 would allow us to generalize (1.7) to this context.

3.5. Fano varieties. A terminal Fano variety X is a normal projective variety with terminal singularities such that $-K_X$ is ample. (We have similar definitions for canonical singularities, log terminal singularities, etc.) These varieties naturally arise in the context of the minimal model program, which predicts that if Y is a variety with $\kappa(Y) < 0$, then there is a finite sequence of flips and divisorial contractions $Y \dashrightarrow Y'$ and a projective morphism $f : Y' \to Z$ whose general fiber is a terminal Fano variety (of dimension > 0). Therefore, one can think of terminal Fano varieties with Picard number one as the building blocks for smooth projective varieties with negative Kodaira dimension $\kappa(Y) < 0$.

If $\dim X = 1$, there is only one terminal Fano variety: the projective line $\mathbb{P}^1$. If $\dim X = 2$, terminal Fano varieties are known as Del Pezzo surfaces (a terminal surface is necessarily smooth). There are ten families of such surfaces. In higher dimensions, one expects a similar result to hold. The following fundamental result (cf. [Nadel90], [Nadel91], [Campana91], [Campana92], [KMM92a], [KMM92b]) shows that at least for smooth Fano varieties, this is the case:

Theorem 3.10. *Let $n \in \mathbb{Z}_{>0}$. Then there are only finitely many families of n-dimensional smooth projective Fano varieties.*

The proof of this Theorem is based on the study of the properties of rational curves on these manifolds. When X has singularities, then the behavior of rational curves on X is more subtle. Nevertheless we have the following conjecture known as the BAB (or Borisov-Alexeev-Borisov) Conjecture.

Conjecture 3.11. *Let $n \in \mathbb{Z}_{>0}$. Then there are only finitely many families of canonical $\mathbb{Q}$-factorial Fano varieties.*

Remark 3.12. *The above conjecture is already interesting for Fano varieties of Picard number 1.*

One also expects a similar conjecture for ϵ-log terminal Fano varieties (not necessarily $\mathbb{Q}$-factorial with arbitrary Picard number). Recall that if $\epsilon > 0$, then X is ϵ-log terminal if for any log resolution $f : X' \to X$, we have $K_{X'} = f^*K_X + \sum a_i E_i$ where $a_i > \epsilon - 1$. The example of cones over a rational curve of degree n show that the ϵ-log terminal condition is indeed necessary.

Conjecture 3.11 is known for canonical Fano varieties of dimension ≤ 3 (in characteristic zero) of arbitrary Picard number cf. [Kawamata92] and [KMMT00]; for toric varieties [BB92] and for spherical varieties [AB04].

A positive answer to Conjecture 3.11 would have profound implications on the birational geometry of higher dimensional projective varieties. In particular (3.11) is related to the famous conjectures on the ACC for mld's, the ACC for log canonical thresholds and the termination of flips.

The techniques for the study of varieties of positive Kodaira dimension (that we have described above) do not readily apply to this context. However we would like to mention [McKernan03] for a related approach and [HM10b] for one possible connection showing that it is possible that results for varieties of log general type may be useful in the study of log-Fano varieties.

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