Name __________________________ Class Time _____

Math 2280 Extra Credit Problems
Chapter 5
S2019

Submitted work. Please submit one stapled package per chapter. Kindly label problems **Extra Credit**. Label each problem with its corresponding problem number, e.g., Xc5.1-8. Please attach this printed sheet to simplify your work.

Problem Xc5.0-1. (Eigenpairs of a Matrix A)

(a) Let $A = \begin{pmatrix} 9 & -10 \\ 2 & 0 \end{pmatrix}$. Find the eigenpairs of A. Then report eigenpair packages P and D such that $AP = PD$.

(b) Let $A = \begin{pmatrix} 5 & -6 & 3 \\ 6 & -7 & 3 \\ 6 & -6 & 2 \end{pmatrix}$. Find the eigenpairs of A. Then report eigenpair packages P and D such that $AP = PD$.

(c) Let $A = \begin{pmatrix} 0 & -6 \\ 24 & 0 \end{pmatrix}$. Find the complex eigenpairs of A. Then report eigenpair packages P and D such that $AP = PD$.

(d) Let $A = \begin{pmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{pmatrix}$. Find the eigenpairs of A. Then report eigenpair packages P and D such that $AP = PD$.

Check the answer by hand, expanding both products AP and PD, finally showing equality.

Problem Xc5.0-36. (Eigenvalues of band matrices)

Find the eigenvalues of the matrix A below without the aid of computers.

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem Xc5.0-18. (Fourier's model for a 3×3)

Assume Fourier's model for a certain matrix A:

$$A\left(c_1\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = 3c_1\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Fourier's model (the above equation) is assumed valid for all constants c_1, c_2, c_3. Find A explicitly from $AP = PD$. Check your answer by finding the eigenpairs of A.

Problem Xc5.0-28. (Eigenpairs and Diagonalization of a 4×4)

Determine the eigenpairs of A below. If diagonalizable, then report eigenpair packages P and D such that $AP = PD$.

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 13 \end{pmatrix}$$

Problem Xc5.1-14. (Particular solution)

(a) Find the constants c_1, c_2 in the general solution

$$\mathbf{x}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

satisfying the initial conditions $x_1(0) = 4$, $x_2(0) = -1$.

(b) Find the matrix A in the equation $\mathbf{x}' = A\mathbf{x}$. Use the formula $AP = PD$ and Fourier's model for A, which is given implicitly in (a) above, and explicitly as

$$A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2$$

where c_1, c_2 are arbitrary constants and $(\lambda_1, \mathbf{v}_1)$, $(\lambda_2, \mathbf{v}_2)$ are the eigenpairs of the 2×2 matrix A.

Problem Xc5.2-8. (Eigenanalysis method 2×2)

(a) Find λ_1, λ_2, $\mathbf{v}_1$, $\mathbf{v}_2$ in Fourier's model $A\,(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2$ for

$$A = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}.$$

(b) Display the general solution of $\mathbf{x}' = A\mathbf{x}$.

Problem Xc5.2-20. (Eigenanalysis method 3×3)

(a) Find λ_1, λ_2, λ_3, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in Fourier's model $A\,(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3$ for

$$A = \begin{pmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{pmatrix}.$$

(b) Display the general solution of $\mathbf{x}' = A\mathbf{x}$.

Problem Xc5.2-30. (Brine Tanks)

Consider two brine tanks satisfying the equations

$$x_1'(t) = -k_1x_1 + k_2x_2, \quad x_2' = k_1x_1 - k_2x_2.$$

Assume $r = 10$ gallons per minute, $k_1 = r/V_1$, $k_2 = r/V_2$, $x_1(0) = 30$ and $x_2(0) = 0$. Let the tanks have volumes $V_1 = 50$ and $V_2 = 25$ gallons. Solve for $x_1(t)$ and $x_2(t)$.

Problem Xc5.2-40. (Eigenanalysis method 4×4)

Display (a) Fourier's model and (b) the general solution of $\mathbf{x}' = A\mathbf{x}$ for the 4×4 matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -21 & -5 & -27 & -9 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -16 & -4 \end{pmatrix}.$$

Problem Xc5.5-4. (Fundamental Matrix)

This problem is double credit, to match the effort required. Consider the 2×2 vector-matrix differential equation

$$\mathbf{u}' = A\mathbf{u}, \quad A = \begin{pmatrix} 2 & -5 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Complete all parts below.

(a) *Cayley-Hamilton method.* Compute the characteristic equation $\det(A - \lambda I) = 0$. Find two atoms from the roots of this equation. Then $x(t)$ is a linear combination of these atoms. The first equation $x' = 2x - 5y$ can be solved for y to find the second answer. Construct a fundamental matrix Φ from these scalar answers.

(b) *Eigenanalysis method.* Find the eigenpairs $(\lambda_1, \mathbf{v}_1), (\lambda_2, \mathbf{v}_2)$ of A. Let Φ have columns $e^{\lambda_1 t}\mathbf{v}_1$, $e^{\lambda_2 t}\mathbf{v}_2$. Explain why Φ is a fundamental matrix.

(c) *Putzer's formula.* Find e^{At} from the formula

$$e^{At} = e^{\lambda_1 t} T + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I).$$

If the eigenvalues are complex, then e^{At} is the real part of the right side. If $\lambda_1 = \lambda_2$, then e^{At} is the limit of the right side as $\lambda_2 \to \lambda_1$ (use L'Hopital's rule).

(d) Report $e^{At} = \Phi(t)\Phi(0)^{-1}$, using the answer for Φ from part (a) or (b). Check your answer against the one in part (c).

Problem Xc5.5-12. (Putzer's Method)

The exponential matrix e^{At} can be found in the 2×2 case from Putzer's formula

$$e^{At} = e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I).$$

If the roots λ_1, λ_2 of $\det(A - \lambda I) = 0$ are equal, then compute the Newton quotient factor by L'Hopital's rule, limiting $\lambda_2 \to \lambda_1$ [λ_1, t fixed]. If the roots are complex, then take the real part of the right side of the equation.

Compute e^{At} from Putzer's formula for the following cases.

(a) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Answer $e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$.

(b) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$.

(d) $A = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$.

Problem Xc5.5-38. (Laplace's Resolvent Method)

The exponential matrix e^{At} can be found from the Laplace resolvent formula for the problem $\Phi' = A\Phi$, $\Phi(0) = I$:

$$\mathcal{L}(\Phi(t)) = (sI - A)^{-1}\Phi(0) = (sI - A)^{-1}.$$

For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ gives $\mathcal{L}(e^{At}) = \begin{pmatrix} s-1 & 0 \\ 0 & s-2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{pmatrix} = \begin{pmatrix} \mathcal{L}(e^t) & 0 \\ 0 & \mathcal{L}(e^{2t}) \end{pmatrix}$, which implies $e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$.

Compute $\Phi(t) = e^{At}$ using the resolvent formula for the following cases.

(a) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$.

Problem Xc5.6-4. (Variation of Parameters)

Use the variation of parameters formula $\mathbf{u}_p(t) = e^{At}\int e^{-At}\mathbf{f}(t)dt$ to find a particular solution of the given system. Please use `maple` to do the indicated integration, following the example below.

(a) $\mathbf{u}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(b) $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} e^t \\ 1 \end{pmatrix}$.

Example: Solve for $\mathbf{u}_p(t)$: $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

```
with(LinearAlgebra):
A:=Matrix([[0,1],[1,0]]);
f:=t->Vector([1,0]);
expAt:=t->MatrixExponential(A,t);
integral:=Map(g->int(g,t),expAt(-t).f(t));
up:=simplify(expAt(t).integral);
```

Problem Xc5.6-19. (Initial Value Problem)

Solve the given initial value problem using a computer algebra system. Follow the example given below.

(a) $\mathbf{u}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{u}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(b) $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} e^t \\ 1 \end{pmatrix}$, $\mathbf{u}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Example: Solve for $\mathbf{u}(t)$: $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{u} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $\mathbf{u}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. The answer is $\mathbf{u} = \begin{pmatrix} -e^{-t} \\ e^{-t} - 1 \end{pmatrix}$.

```
with(LinearAlgebra):
A:=Matrix([[0,1],[1,0]]);
f:=t->Vector([1,0]);
expAt:=t->MatrixExponential(A,t);
integral:=Map(g->int(g,t=0..t),expAt(-t).f(t));
up:=unapply(expAt(t).integral,t):
u0:=Vector([-1,0]);
uh:=t->expAt(t).(u0-up(0));
u:=simplify(uh(t)+up(t));
```

End of extra credit problems chapter 5.