$\boxed{\text{Solutions to Review Problem for Exam 1}}$

① $x' = x^2 - x^4$ Setting $f(x) = x^2 - x^4$, this is an autonomous equation $x' = f(x)$. The equilibrium solutions are critical points, solutions of $f(x) = 0$

$$f(x) = x^2 - x^4 = x^2(1-x^2) = x^2(1-x)(1+x)$$

Thus the equilibrium solutions are $\boxed{x = 0, \ x = -1, \ x = 1.}$

We check the sign of f on the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$

On $(-\infty, -1)$ $f(x) < 0$ $\quad$ ($x^2(1-x)(1+x)$ = pos. · pos. neg.)

On $(-1, 0)$ $f(x) > 0$ $\quad$ ($x^2(1-x)(1+x)$ = pos · pos. · pos.)

(On $(0, 1)$ $f(x) > 0$ $\quad$ ($x^2(1-x)(1+x)$ = pos. pos. pos.)

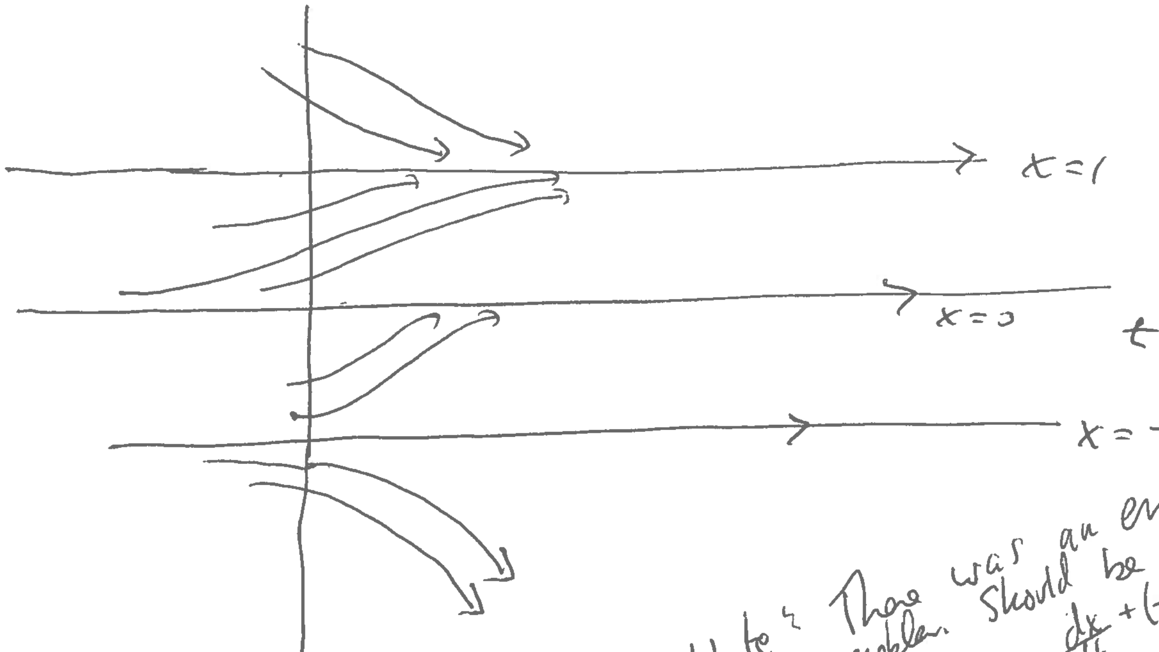
On $(1, \infty)$ $f(x) < 0$ $\quad$ $x^2(1-x)(1+x)$ = pos. · neg. pos.

$\Rightarrow$ $x = -1$ is <u>unstable</u>

$x = 0$ is <u>unstable</u>

$x = 1$ is stable (asymptotically stable

Solution Curves



Note: There was an error in this problem. Should be $\frac{dx}{dt} + \left(\frac{3}{60-t}\right)x = 2$

(2) $\frac{dx}{dt}$ = rate at which salt enters − rate at which salt leaves.

(a) The rate at which salt enters is $\frac{1\,lb}{gal} \cdot \frac{2\,gal}{min} = 2\,\frac{lb}{min}$

The rate at which salt leaves:

concentration · vol rate leaving ← vol leaves at rate of 3 gal/min

concentration at time $t = \frac{X(t)}{V(t)}$ where $V(t)$ = vol.

But net volume changes at a net rate of −1 gal/min since enters at 2 gal/min, leaves at 3 gal/min. Therefore $V(t) = 60 - t$ gallons (starting at 60 gal. when $t = 0$).

⇒ rate at which salt leaves is

$\frac{x}{60-t} \cdot 3$. Thus

$$\frac{dx}{dt} = 2 - \frac{3x}{60-t} \implies \frac{dx}{dt} + \frac{3x}{60-t} = 2$$

(b) This is a first order linear eq. Using integrating factor: $\rho(t) = e^{\int P\,dt} = e^{\int \frac{3}{60-t}\,dt}$ $\qquad P(t) = \frac{3}{60-t}$

$$\int \frac{3}{60-t}\,dt = -3\int \frac{dt}{t-60} = -3\ln|t-60|$$

$$\rho(t) = e^{-3\ln|t-60|} = e^{\ln|t-60|^{-3}} = |t-60|^{-3} = (60-t)^{-3}$$

since $t \leq 0$ (when $t = 60$ tank is empty)

$$(60-t)^{-3}\frac{dx}{dt} + (60-t)^{-3}\frac{3x}{60-t} = 2(60-t)^{-3}$$

$$(60-t)^{-3}\frac{dx}{dt} + (60-t)^{-4}\,3x = 2(60-t)^{-3}$$

$$\left((60-t)^{-3}x\right)' = 2(60-t)^{-3} \quad \text{Integrate}$$

$$(60-t)^{-3}x = 2\int (60-t)^{-3}\,dt = \qquad u = 60-t,\ du = -dt$$

$$= -2\frac{(60-t)^{-2}}{-2} + c = (60-t)^{-2} + c$$

$$\Rightarrow (60-t)^3(60-t)^{-3}x = (60-t)^3\left((60-t)^{-2} + c\right)$$

$$x = 60-t + c(60-t)^3$$

$$x(0) = 0 = 60 + c60^3 \Rightarrow \quad c60^3 = -60$$

$$c = -\frac{1}{60^2} = \frac{-1}{3600}$$

$$\boxed{x(t) = 60-t - \frac{1}{3600}(60-t)^3}$$

③ $2y^{(3)} - 2y'' + 13y' = 0$

Characteristic equation is

$$2r^3 - 2r^2 + 13r = 0$$

$$\Rightarrow \quad r(2r^2 - 2r + 13) = 0$$

roots are: $r = 0$, $r = \frac{2 \pm \sqrt{4 - 4 \cdot 2 \cdot 13}}{4} = \frac{2 \pm 2\sqrt{1-26}}{4}$

$\Rightarrow \quad r = 0, \quad r = \frac{1}{2} \pm \frac{5i}{2}$

Corresponding solutions are $y_1 = 1$

$y_2 = e^{\frac{1}{2}x} \cos \frac{5}{2}x$

$y_3 = e^{\frac{1}{2}x} \sin \frac{5}{2}x$

$\Rightarrow$ General Solution is

$$y = C_1 + C_2 e^{\frac{1}{2}x} \cos \frac{5}{2}x + C_3 e^{\frac{1}{2}x} \sin \frac{5}{2}x$$

(4) $y'' + 5y' - 6y = f(x)$

a. $f(x) = 0$: Characteristic equation is

$$r^2 + 5r - 6 = 0$$

$$\Rightarrow (r-1)(r+6) = 0$$

roots : $r = 1, -6$.

General Soln. of homogeneous equation is

$$\boxed{y = c_1 e^{x} + c_2 e^{-6x}}$$

(b) $f(x) = 10^{-x}$. $y_p(x) = -e^{-x}$.

$y_p' = e^{-x}$, $y_p'' = -e^{-x}$.

$\Rightarrow y_p'' + 5y_p' - 6y_p = -e^{-x} + 5e^{-x} + 6e^{-x} = 10e^{-x}$

$\Rightarrow y_p$ is a particular solution.

(c) $\cancel{y(x) = c_1 e^{x} + c_2 e^{-6x} +}$

$y(x) = y_c(x) + y_p(x)$ where y_c is a solution of the homogeneous equation. Therefore, combining a, b:

$y(x) = c_1 e^{x} + c_2 e^{-6x} - e^{-x}$ for some constants c_1 and c_2.

We compute $y'(x)$ and plug in the initial conditions at $x=0$:

$$y'(x) = c_1 e^{x} - 6c_2 e^{-6x} + e^{-x}$$

$$y(0) = c_1 + c_2 - 1 = 2 \quad \Rightarrow \quad c_1 + c_2 = 3$$

$$y'(0) = c_1 - 6c_2 + 1 = 2 \qquad c_1 - 6c_2 = 1$$

~~Subtracting~~ Subtracting 2nd eq. from first gives:

$$7c_2 = 2 \Rightarrow c_2 = 2/7$$

$$\Rightarrow c_1 + 2/7 - 1 = 2 \Rightarrow c_1 = \frac{19}{7}$$

$$\boxed{y(x) = \frac{19}{7} e^{x} + \frac{2}{7} e^{-6x} - e^{-x}}$$

(5) $\dfrac{dy}{dx} = 3x^2(y^2+1)$

(a) The equation is of the form $\dfrac{dy}{dx} = f(x,y)$ with $f(x,y) = 3x^2(y^2+1)$, and f as well as $\dfrac{\partial f}{\partial y} = 2y \cdot 3x^2$ are continuous in a rectangle containing $(0,0)$ (indeed, containing any point). Therefore, there is a unique solution on some interval containing 0, by the Existence and Uniqueness Theorem.

(b) The equation is separable:

$$\frac{1}{y^2+1}\frac{dy}{dx} = 3x^2 \implies$$

$$\int \frac{1}{y^2+1}\frac{dy}{dx}\,dx = \int 3x^2\,dx$$

$$\int \frac{1}{y^2+1}\,dy = x^3 + C \qquad \text{some constant } C$$

$$\tan^{-1}(y) = x^3 + C$$

$$\implies \tan(\tan^{-1}(y)) = \tan(x^3+C)$$

$$y = \tan(x^3+C)$$

$$y(0)=1 \implies \tan C = 1 \qquad \text{solving for } C$$

$$\implies C = \tan^{-1} 1 = \frac{\pi}{4}$$

$$\implies \boxed{y = \tan\left(x^3+\frac{\pi}{4}\right)}$$

Note: there are other constants: $C = \frac{\pi}{4} + n\pi$ also satisfies $\tan C = \tan\left(\frac{\pi}{4}+n\pi\right) = 1$, but these yield the same solution since $\tan x$ has period π.