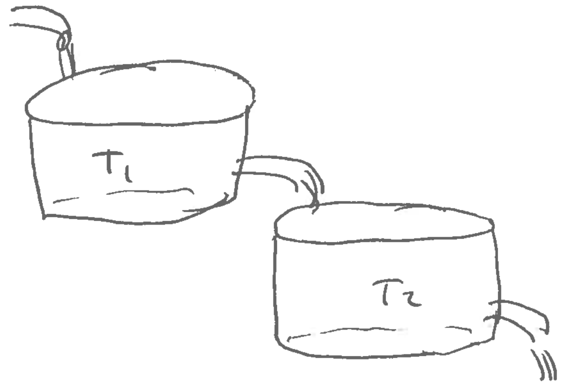
Solutions to Review Problems for the Final Exam

1.



$V_1 = 50$

$V_2 = 150$

$r =$ flow rate $= 5$

$\left.\begin{aligned} c_1 &= \frac{x}{V_1} = \frac{x}{50} \\ c_2 &= \frac{y}{V_2} = \frac{y}{150} \end{aligned}\right\}$ concentrations

a. $x' =$ rate leaving $= -\frac{x}{50} \cdot 5 = -0.1x$

$y' =$ rate entering $-$ rate leaving $= 0.1x - \frac{y}{100} \cdot 5 = 0.1x - 0.05y$

i.e. $x' = -0.1x$
$y' = 0.1x - 0.05y$

b. Using the eigenvalue method:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -0.1 & 0 \\ 0.1 & -0.05 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Lower triangular matrix $\Rightarrow$ eigenvalues are $\lambda_1 = -0.1$, $\lambda_2 = -0.05$.

Eigenvector $\vec{v}_1$: $\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0.1 & 0.05 & 0 \end{array}\right) \rightarrow 0.1x = -0.05y$

$x = -0.5y$

setting $y = 2 \Rightarrow \vec{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Eigenvector $\vec{v}_2$: ($\lambda_2 = -0.05$) $\left(\begin{array}{cc|c} -0.05 & 0 & 0 \\ 0.1 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow \begin{aligned} x_1 &= 0 \\ x_2 &= x_2 \end{aligned}$

Setting $x_2 = 1$ we get $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

general soln:

$$\vec{x}(t) = c_1 e^{-0.1t}\begin{pmatrix}-1\\2\end{pmatrix} + c_2 e^{-0.05t}\begin{pmatrix}0\\1\end{pmatrix}$$

$$\vec{x}(0) = c_1\begin{pmatrix}-1\\2\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix}$$

$$\left(\begin{array}{cc|c}-1 & 0 & 3\\ 2 & 1 & 0\end{array}\right) \to \left(\begin{array}{cc|c}-1 & 0 & 3\\ 0 & 1 & 6\end{array}\right) \to \left(\begin{array}{cc|c}1 & 0 & -3\\ 0 & 1 & 6\end{array}\right) \qquad \begin{array}{l}c_1 = -3\\ c_2 = 6\end{array}$$

$$\boxed{\vec{x}(t) = -3e^{-0.1t}\begin{pmatrix}-1\\2\end{pmatrix} + 6e^{-0.05t}\begin{pmatrix}0\\1\end{pmatrix}}$$

or

$$\boxed{\begin{array}{l}x(t) = 3e^{-0.1t}\\ y(t) = -6e^{-0.1t} + 6e^{-0.05t}\end{array}}$$

2. $\vec{x}' = \vec{F}(\vec{x})$ where $\vec{x} = \begin{pmatrix}x\\y\end{pmatrix}$, $F(\vec{x}) = \begin{pmatrix}x^2 - 2xy + y^2 - 1\\ 4 - y^2\end{pmatrix}$

Critical points: $\vec{F}(\vec{x}) = 0$

$$\begin{array}{l}x^2 - 2xy + y^2 - 1 = 0\\ 4 - y^2 = 0\end{array} \Rightarrow y = \pm 2$$

$y = 2$ $\quad x^2 - 4x + 3 = 0 \Rightarrow (x-3)(x-1) = 0 \quad x = 1, 3$

$\boxed{(1,2),\ (3,2)}$

$y = -2$: $\quad x^2 + 4x + 3 = 0 \Rightarrow (x+3)(x+1) = 0 \quad x = -1, -3$

$\boxed{(-1,-2),\ (-3,-2)}$

$$D\vec{F}(x,y) = \begin{pmatrix}2x - 2y & -2x + 2y\\ 0 & -2y\end{pmatrix}$$

$$D\vec{F}(1,2) = \begin{pmatrix} 2-4 & -2+4 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 0 & -4 \end{pmatrix}$$

eigenvalues are $-2, -4$, distinct real and negative.

(1,2) Asymptotically stable node

$$DF(3,2) = \begin{pmatrix} 6-4 & -6+4 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 0 & -4 \end{pmatrix}$$

Eigenvalues are $2, -4$, real, opposite sign

$\Rightarrow$ (3,2) unstable saddle point

$$D\vec{F}(-1,-2) = \begin{pmatrix} -2+4 & 2-4 \\ 0 & +4 \end{pmatrix} = \begin{pmatrix} +2 & -2 \\ 0 & 4 \end{pmatrix}$$

eigenvalues $+2, +4$ pos. same sign

(−1, −2) unstable ~~saddle~~ node

$$D\vec{F}(-3,-2) = \begin{pmatrix} -6+4 & 6-4 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 0 & 4 \end{pmatrix}$$

eigenvalues are $-2, 4$ $\Rightarrow$ (−3, −2) unstable saddle

real, opposite sign

$y^{(4)} + 4y^{(3)} + 8y'' + 16y' + 16y = t + 2$

3. Characteristic equation is:

a. $p(r) = r^4 + 4r^3 + 8r^2 + 16r + 16 = 0$

Since -2 is a root $(r+2)$ is a factor. Use long division to find other factors:

$$
\begin{array}{r|l}
 & r^3 + 2r^2 + 4r + 8 \\
\hline
r+2 & r^4 + 4r^3 + 8r^2 + 16r + 16 \\
 & r^4 + 2r^3 \\
\hline
 & 2r^3 + 8r^2 + 16r + 16 \\
 & 2r^3 + 4r^2 \\
\hline
 & 4r^2 + 16r + 16 \\
 & 4r^2 + 8r \\
\hline
 & 8r + 16 \\
 & 8r + 16 \\
\hline
 & 0
\end{array}
$$

$p(r) = (r+2)(r^3 + 2r^2 + 4r + 8)$

by inspection -2 is also a root of $r^3 + 2r^2 + 4r + 8$

divide again

$$
\begin{array}{r|l}
 & r^2 + 4 \\
\hline
r+2 & r^3 + 2r^2 + 4r + 8 \\
 & r^3 + 2r^2 \\
\hline
 & 4r + 8
\end{array}
$$

$\Rightarrow p(r) = (r+2)(r+2)(r^2+4) = 0$

roots: $r = -2$ multiplicity 2

$r = 2i$

$r = -2i$

Solutions of homogeneous eq. corresponding to $r=-2$:

$y_1 = e^{-2x}$, $y^2 = xe^{-2x}$.

Corresponding to $r = \pm 2i$: $y_3 = \cos 2x$, $y_4 = \sin 2x$

General soln. of homog. eq:

$$\boxed{y = c_1 e^{-2x} + c_2 x e^{-2x} + c_3 \cos 2x + c_4 \sin 2x}$$

b. A particular solution has form: $y_p = At + B$

$y_p' = A$, $y_p'' = y_p''' = y_p'''' = 0$. Plug into equation:

$16 \cdot A + 16(At + B) = t + 2$

$16A + 16B + 16At = t + 2$ equate coefficients

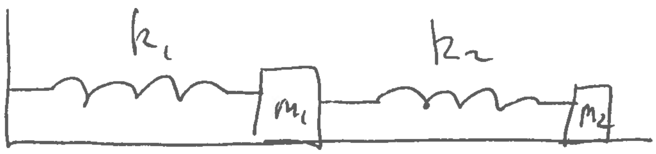
$$\begin{aligned} 16A &= 1 \\ 16A + 16B &= 2 \end{aligned} \Rightarrow \begin{aligned} A &= 1/16 \\ 16 \cdot \tfrac{1}{16} + 16B &= 2 \end{aligned} \Rightarrow 16B = 1 \Rightarrow B = 1/16$$

$y_p = \frac{1}{16} t + \frac{1}{16}$

General Solution:

$$\boxed{y = c_1 e^{-2x} + c_2 x e^{-2x} + c_3 \cos 2x + c_4 \sin 2x + \frac{1}{16} t + \frac{1}{16}}$$

4.



$m_1 = 2, \quad m_2 = 1/2$

$k_1 = 75, \quad k_2 = 25$

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad K = \begin{pmatrix} -(k_1+k_2) & k_2 \\ k_2 & -k_2 \end{pmatrix} \quad \text{(from class)}$$

Should be $M\vec{x}'' = K\vec{x}$

$$= \begin{pmatrix} -100 & 25 \\ 25 & -25 \end{pmatrix}$$

$$M\vec{x}'' = K\vec{x} \Rightarrow \boxed{\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \vec{x}'' = \begin{pmatrix} -100 & 25 \\ 25 & -25 \end{pmatrix} \vec{x}}$$

Multiply by M^{-1} to put in form $\vec{x}'' + A\vec{x} = 0$

$$A = M^{-1}K = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -100 & 25 \\ 25 & -25 \end{pmatrix} = \begin{pmatrix} -50 & 12.5 \\ 50 & -50 \end{pmatrix}$$

Eigenvalues of A: $(-50-\lambda)(-50-\lambda) - 625 = 0$

$$(-50-\lambda)^2 = 625 = (50+\lambda)^2$$

$$50 + \lambda = \pm 25$$

$$\lambda = \pm 25 - 50$$

$$\lambda_1 = -25, \quad \lambda_2 = -75$$

Fundamental frequencies are ω_i where $\lambda_i = -\omega_i^2$, therefore

$\boxed{\omega_1 = 5}$ and $\omega_2 = \sqrt{75} = 5\sqrt{3}$

Eigenvectors of A:

$\vec{v}_1 \ (\lambda_1 = -25)$: $\left(\begin{array}{cc|c} -25 & 12.5 & 0 \\ 50 & -25 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

or $\left(\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$$x_1 = \frac{1}{2} x_2 \qquad \text{set } x_2 = 2 \implies \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$x_2 = x_2$$

$\vec{v}_2 : (\lambda_2 = -75)$ $\begin{pmatrix} 25 & 12.5 \\ 50 & 25 \end{pmatrix} \to \left(\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$

$$x_1 = -\frac{1}{2} x_2 \qquad \text{set } x_2 = 2 \implies \vec{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
$$x_2 = x_2$$

General solution is

$$\vec{x}(t) = \left(a_1 \cos 5t + b_1 \sin 5t\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \left(a_2 \cos 5\sqrt{3}t + b_2 \sin 5\sqrt{3}t\right)\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

c. $M\vec{x}'' = K\vec{x} + \vec{F}$ when $\vec{F} = \begin{pmatrix} 0 \\ 100\cos 10t \end{pmatrix}$

$$\Rightarrow \quad \vec{x}'' = A\vec{x} + M^{-1}\vec{F}$$

$$\vec{x}'' = A\vec{x} + \begin{pmatrix} 0 \\ 200\cos 10t \end{pmatrix}$$

$\vec{x}_p$ has the form: $\cos 10t\,\vec{a}$ some vector $\vec{a}$ to be determined. $\vec{x}_p'' = -100\cos 10t\,\vec{a}$. Plug into equation.

$$-100\,\cancel{\cos 10t}\,\vec{a} = \cancel{\cos 10t}\,A\vec{a} + \cancel{\cos 10t}\begin{pmatrix} 0 \\ 200 \end{pmatrix}$$

$$\Rightarrow \quad A\vec{a} + 100I\,\vec{a} = -\begin{pmatrix} 0 \\ 200 \end{pmatrix}$$

$$(A + 100I)\,\vec{a} = \begin{pmatrix} 0 \\ -200 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 50 & 12.5 & 0 \\ 50 & 50 & -200 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & -4 \\ 50 & 12.5 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 1 & -4 \\ 0 & -37.5 & 200 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & -4 \\ 0 & 1 & -\frac{16}{3} \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 0 & 4/3 \\ 0 & 1 & -16/3 \end{array}\right)$$

$$\vec{x}_p = \cos 10t \begin{pmatrix} 4/3 \\ -16/3 \end{pmatrix}$$

General Solution is then

$$\vec{x}(t) = (a_1 \cos 5t + b_1 \sin 5t)\begin{pmatrix}1\\2\end{pmatrix} + (a_2 \cos 5\sqrt{3}t + b_2 \sin 5\sqrt{3}t)\begin{pmatrix}-1\\2\end{pmatrix} + \cos 10t \begin{pmatrix}4/3\\-16/3\end{pmatrix}$$

$$\vec{x}'(t) = (-5a_1 \sin 5t + 5b_1 \cos 5t)\begin{pmatrix}1\\2\end{pmatrix} + (-5\sqrt{3}a_2 \sin 5\sqrt{3}t + 5\sqrt{3}b_2 \cos 5\sqrt{3}t)\begin{pmatrix}-1\\2\end{pmatrix} - 10 \sin 10t \begin{pmatrix}4/3\\-16/3\end{pmatrix}$$

$$\vec{x}(0) = a_1\begin{pmatrix}1\\2\end{pmatrix} + a_2\begin{pmatrix}-1\\2\end{pmatrix} + \begin{pmatrix}4/3\\-16/3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\vec{x}'(0) = 5b_1\begin{pmatrix}1\\2\end{pmatrix} + 5\sqrt{3}b_2\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

a_1, a_2:
$$\left(\begin{array}{cc|c}1 & -1 & -4/3\\2 & 2 & 16/3\end{array}\right) \to \left(\begin{array}{cc|c}1 & -1 & -4/3\\0 & 4 & 8\end{array}\right) \to \left(\begin{array}{cc|c}1 & -1 & -4/3\\0 & 1 & 2\end{array}\right)$$

$$\to \left(\begin{array}{cc|c}1 & 0 & 2/3\\0 & 1 & 2\end{array}\right) \qquad \begin{array}{l}a_1 = 2/3\\a_2 = 2\end{array}$$

b_1, b_2:
$$b_1\begin{pmatrix}1\\2\end{pmatrix} + \sqrt{3}\begin{pmatrix}-1\\2\end{pmatrix}b_2 = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \left(\begin{array}{cc|c}1 & -\sqrt{3} & 0\\2 & 2\sqrt{3} & 0\end{array}\right) \to b_1 = b_2 = 0.$$

$$\boxed{\vec{x}(t) = \frac{2}{3}\cos 5t\begin{pmatrix}1\\2\end{pmatrix} + 2\cos 5\sqrt{3}t\begin{pmatrix}-1\\2\end{pmatrix} + \cos 10t\begin{pmatrix}4/3\\-16/3\end{pmatrix}}$$

5. $f(x) = \begin{cases} 1-x & 0 \le x \le 1 \\ 0 & 1 \le x \le 2 \end{cases}$ $\qquad L = 2$

a. $a_0 = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}\int_0^1 1-x\,dx = \frac{1}{2}\left(x - \frac{x^2}{2}\right)\Big|_0^1 = \frac{1}{4}$

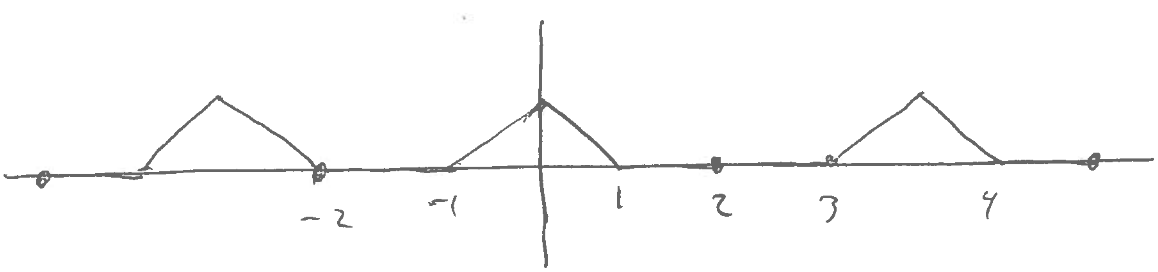
$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi}{2}x\,dx = \int_0^1 (1-x)\cos\frac{n\pi}{2}x\,dx = \int_0^1 \cos\frac{n\pi}{2}x\,dx - \int_0^1 x\cos\frac{n\pi}{2}x\,dx$

$= \frac{2}{n\pi}\sin\frac{n\pi}{2}x\Big|_0^1 - \left(x\cdot\frac{2}{n\pi}\sin\frac{n\pi}{2}x\Big|_0^1 - \frac{2}{n\pi}\int_0^1 \sin\frac{n\pi}{2}x\,dx\right)$

$= \cancel{\frac{2}{n\pi}\sin\frac{n\pi}{2}} - \left(\cancel{\frac{2}{n\pi}\sin\frac{n\pi}{2}} + \frac{2}{n\pi}\cdot\frac{2}{n\pi}\cos\frac{n\pi}{2}x\Big|_0^1\right)$

$= \frac{4}{\pi^2 n^2}\left(1 - \cos\frac{n\pi}{2}\right)$

$$f(x) = \frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\left(1 - \cos\frac{n\pi}{2}\right)\cos\frac{n\pi}{2}x$$



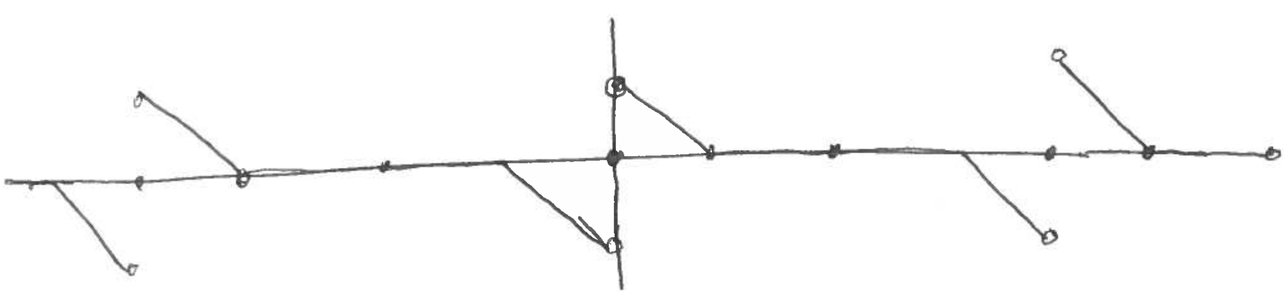
b. $b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi}{2}x\,dx = \int_0^1 (1-x)\sin\frac{n\pi}{2}x\,dx = \int_0^1 \sin\frac{n\pi}{2}x\,dx - \int_0^1 x\sin\frac{n\pi}{2}x\,dx$

$$= -\frac{2}{n\pi}\cos\frac{n\pi}{2}x\Big|_0^1 - \left[-x\frac{2}{n\pi}\cos\frac{n\pi}{2}x\Big|_0^1 + \frac{2}{n\pi}\int_0^1 \cos\frac{n\pi}{2}x\,dx\right]$$

$$= \cancel{-\frac{2}{n\pi}\cos\frac{n\pi}{2}} + \frac{2}{n\pi} \cancel{+ \frac{2}{n\pi}\cos\frac{n\pi}{2}} \quad -\frac{2}{n\pi}\cdot\frac{2}{n\pi}\sin\frac{n\pi}{2}\Big|_0^1$$

$$= \frac{2}{n\pi} - \frac{4}{\pi^2 n^2}\sin\frac{n\pi}{2}$$

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2}{n\pi} - \frac{4}{\pi^2 n^2}\sin\frac{n\pi}{2}\right)\sin\frac{n\pi}{2}x$$



6.

$k = 0.1$

a. $u_t = 0.1\, u_{xx}$

$u(0,t) = 0$

$u(2,t) = 0$

$u(x,0) = f(x)$

Using coefficients of the sine series (part b.) from #5

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{n\pi} - \frac{4}{\pi^2 n^2} \sin \frac{n\pi}{2} \right) e^{-\frac{n^2\pi^2 (0.1)}{4} t} \sin \frac{n\pi}{2} x}$$

b. Insulated Ends:

$u_t = 0.1\, u_{xx}$

$u_x(0,t) = 0$

$u_x(2,t) = 0$

$u(x,0) = f(x)$.

Using the coefficient of the cosine series

$$\boxed{u(x,t) = \frac{1}{4} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\left(1 - \cos \frac{n\pi}{2}\right)}{n^2} e^{-\frac{n^2\pi^2 (0.1)}{4} t} \cos \frac{n\pi}{2} x}$$

7. $L = 2$, $a^2 = 6$

a. $u_{tt} = 6u_{xx}$

$u(0,t) = 0$

$u(2,t) = 0$

$u(x,0) = 0.2\sin(\frac{\pi}{2}x) - 0.1\sin\pi x$

b. $f(x)$ is already given by a sine series on interval $0 \le x \le 2$, ~~so~~ with $b_1 = 0.2$, $b_2 = -0.1$ and $b_n = 0$ all $n > 2$. therefore

$$u(x,t) = 0.2\cos\frac{\pi\sqrt{6}}{2}t\sin\frac{\pi}{2}x - 0.1\cos\pi\sqrt{6}\,t\sin\pi x$$

c. Since $f(x) = 0.2\sin\frac{\pi}{2}x - 0.1\sin\pi x$ is 2-periodic and odd, $f(x) = f^*(x)$ (the 2-periodic odd extension of to all $\mathbb{R}$) and therefore

$$u(x,t) = \frac{1}{2}\left(f(x+\sqrt{6}t) + f(x-\sqrt{6}t)\right)$$ or

$$u(x,t) = \frac{1}{2}\Big[0.2\sin\frac{\pi}{2}(x+\sqrt{6}t) - 0.1\sin\pi(x+\sqrt{6}t) + 0.2\sin\frac{\pi}{2}(x-\sqrt{6}t) - 0.1\sin\pi(x-\sqrt{6}t)\Big]$$