Solutions

MATH 2280-001

EXAM 2

Instructions. There are 3 problems on the exam. Put your name on the exam. Calculators or other electronic devices are not allowed, but you are allowed one side of an 8.5 by 11 inch sheet of notes. Show your work for full credit.

1. (33 points) Consider the mass spring equation

$$4x'' + 2x' + \frac{1}{2}x = f(t)$$

a. Explain what x represents, as well as the terms $2x'$ and $\frac{1}{2}x$.

b. Find the general solution of the homogeneous problem.

c. Find the solution satisfying the initial conditions $x(0) = 4$, and $x'(0) = 0$

d. Express the solution found in part c, in the form $Ce^{-pt}cos(\omega_1 t - \alpha)$.

e. If $f(t) = 1 + t + \sin(t)$, what form would a particular solution x_p have)from the method of undetermined coefficients) have?

a. x represents the displacement from equilibrium position
$2x'$ is the force due to friction on the mass
$\frac{1}{2}x$ is the force of the spring on the mass.

b. Characteristic roots : $4r^2 + 2r + \frac{1}{2} = 0 \Rightarrow 8r^2 + 4r + 1 = 0$

$$r = \frac{-4 \pm \sqrt{16-32}}{16} = \frac{-4 \pm 4i}{16} = -\frac{1}{4} \pm \frac{1}{4}i$$

$\Rightarrow$ general solution is $x(t) = c_1 e^{-\frac{1}{4}t}\cos\frac{1}{4}t + c_2 e^{-\frac{1}{4}t}\sin\frac{1}{4}t$

c. $x'(t) = -\frac{1}{4}c_1 e^{-\frac{1}{4}t}\cos\frac{1}{4}t - \frac{1}{4}c_1 e^{-\frac{1}{4}t}\sin\frac{1}{4}t - \frac{1}{4}c_2 e^{-\frac{1}{4}t}\sin\frac{1}{4}t + \frac{1}{4}c_2 e^{-\frac{1}{4}t}\cos\frac{1}{4}t$

$x(0) = c_1 = 4$

$x'(0) = -\frac{1}{4}c_1 + \frac{1}{4}c_2 = 0$

$\Rightarrow$ $c_1 = 4$

$c_1 = c_2 = 4$

$$x(t) = 4e^{-\frac{1}{4}t}\cos\frac{1}{4}t + 4e^{-\frac{1}{4}t}\sin\frac{1}{4}t$$

d. Amplitude $C = \sqrt{c_1^2 + c_2^2} = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$

$\omega_1 = \frac{1}{4}$, $\tan\alpha = \frac{c_2}{c_1} = \frac{4}{4} = 1 \Rightarrow \alpha = \tan^{-1} 1 = \frac{\pi}{4}$

so $\boxed{x(t) = 4\sqrt{2}\cos\left(\frac{1}{4}t - \frac{\pi}{4}\right)}$

e. If $f(t) = 1 + t + \sin t$ then

$$x_p = a_1 + a_2 t + b_1 \cos t + b_2 \sin t$$

for some constants a_1, a_2, b_1, b_2. None of these terms is a solution of the homogeneous equation, thus there is no issue of duplication

2. (34 points) Consider the system of differential equations $\mathbf{x}' = A\mathbf{x}$ where $\mathbf{x} = (x_1\ x_2\ x_3)^T$ and A is the matrix:

$$\begin{pmatrix} 0 & -4 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

8 **a.** Find the eigenvalues of A.

17 **b.** Find the general solution of the system using the eigenvalue method.

9 **c.** Find the particular solution satisfying the initial condition $\mathbf{x}(0) = (3\ 1\ 5\ )^T$

a. $\det\begin{pmatrix} -\lambda & -4 & 0 \\ 1 & -\lambda & -1 \\ 0 & 0 & -1-\lambda \end{pmatrix} = (-1-\lambda)\left[\lambda^2+4\right]$ (expanding across bottom row)

$= -(\lambda+1)(\lambda^2+4) = 0$

$\Rightarrow \lambda = -1,\ \lambda^2+4=0 \Rightarrow \boxed{\lambda = -1,\ 2i,\ -2i}$

b. We find the eigenvector corresponding to $\lambda = -1$:

$$\left(\begin{array}{ccc|c} +1 & -4 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -4 & 0 & 0 \\ 0 & 5 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -4 & 0 & 0 \\ 0 & 1 & -1/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & -4/5 & 0 \\ 0 & 1 & -1/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \begin{array}{l} x_1 = 4/5\, x_3 \\ x_2 = 1/5\, x_3 \\ x_3 = x_3 \end{array}$$ Setting $x_3 = 5 \Rightarrow \vec{v}_1 = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$

The complex eigenvector corresponding to $\lambda = 2i$:

$$\left(\begin{array}{ccc|c} -2i & -4 & 0 & 0 \\ 1 & -2i & -1 & 0 \\ 0 & 0 & -1-2i & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2i & -1 & 0 \\ -2i & -4 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2i & -1 & 0 \\ 0 & 0 & -1-2i & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & -2i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \begin{array}{l} x_1 = 2i\, x_2 \\ x_2 = x_2 \\ x_3 = 0 \end{array}$$ set $x_2 = 1 \Rightarrow \vec{v} = \begin{pmatrix} 2i \\ 1 \\ 0 \end{pmatrix}$

expressing in real + imaginary parts!

$$\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

Complex Solution is $\vec{z}(t) = e^{2it}\,\vec{v} = (\cos 2t + i \sin 2t)\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}\right]$

$$= \left[\cos 2t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \sin 2t \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}\right] + i \left[\cos 2t \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \sin 2t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right]$$

the real + imaginary parts give us independent solution. ~~To get~~ Together with $e^{-t}\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$ we get the general solution:

$$\boxed{\vec{x}(t) = c_1 e^{-t}\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} + c_2\left[\cos 2t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \sin 2t \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}\right] + c_3\left[\cos 2t \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \sin 2t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right]}$$

c. We need to solve for c_1, c_2, c_3:

$$\vec{x}(0) = c_1 \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$$ or

$$\left(\begin{array}{ccc|c} 4 & 0 & 2 & 3 \\ 1 & 1 & 0 & 1 \\ 5 & 0 & 0 & 5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 4 & 0 & 2 & 3 \\ 1 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 \end{array}\right) \to \boxed{c_1 = 1, \quad c_2 = 0, \quad c_3 = -\frac{1}{2}}$$

$$\vec{x}(t) = e^{-t}\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} \cdot -\frac{1}{2}\left[\cos 2t \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \sin 2t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right]$$

3. (33 points) Consider the following linear system of differential equations:

$$x' = x + 3y$$
$$y' = 3x + y$$

a. Express the system in matrix form $\mathbf{x}' = A\mathbf{x}$, and find the eigenvalues, and corresponding eigenvectors of A.

b. What is the general solution (express in vector form)?

c. Classify the critical point $(0,0)$ as a stable or unstable, node, spiral, saddle or center.

d. Sketch the phase diagram of the system. Include the eigenvectors (if real) in your sketch, as well as the trajectories that pass through the points $(1,0), (0,1), (-1,0)$, and $(0-1)$.

a. $\vec{x}' = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \vec{x}$. Eigenvalues: $\det \begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 9 = 0$

$\Rightarrow (1-\lambda)^2 = 9 \Rightarrow 1-\lambda = \pm\sqrt{9} = \pm 3 \Rightarrow \lambda = 1 \pm 3, \boxed{\lambda = -2, 4}$

Eigenvectors: $\lambda_1 = -2$: $\left(\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \begin{array}{l} x_1 + x_2 = 0 \\ x_2 = x_2 \end{array}$

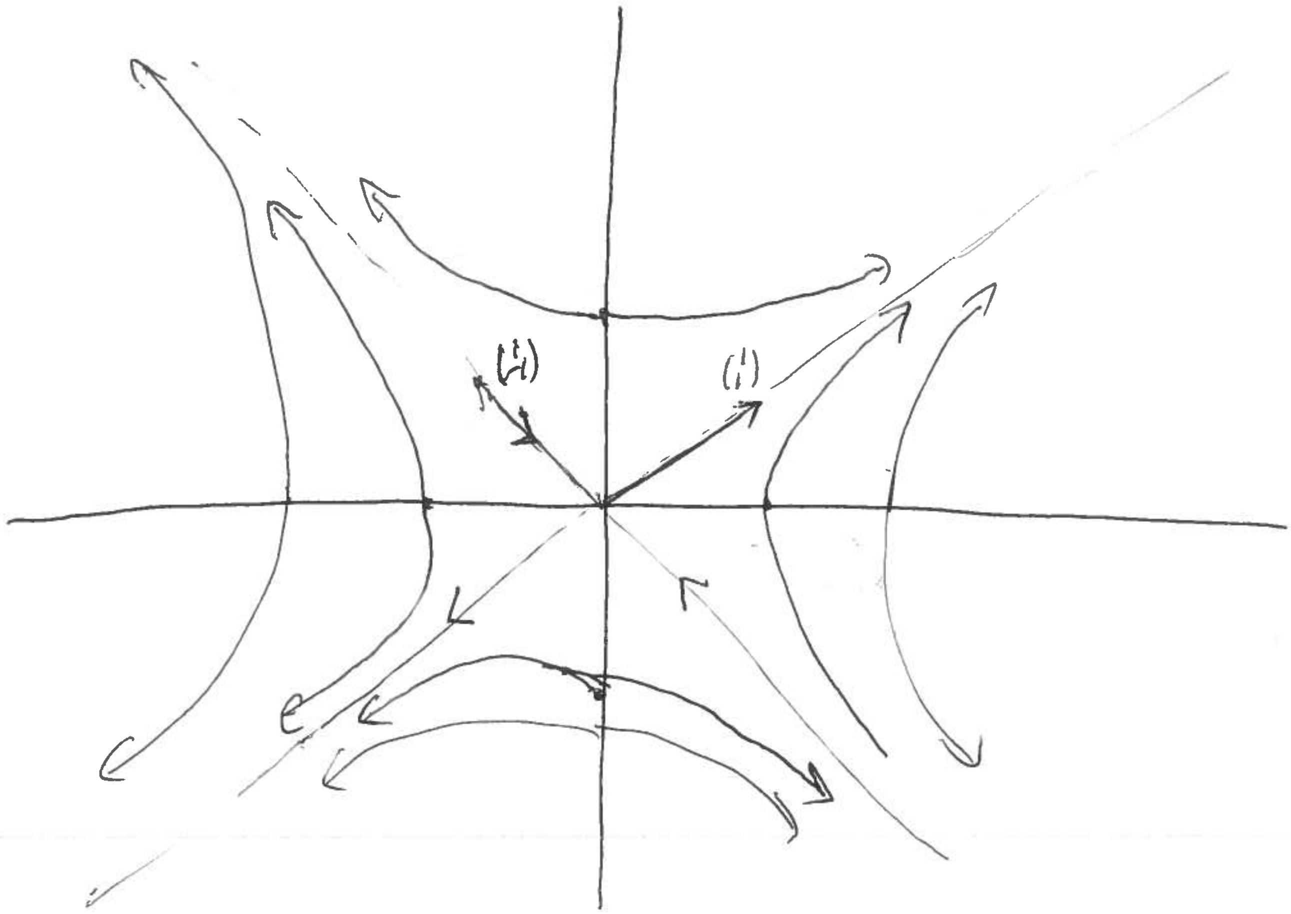
$\begin{array}{l} x_1 = -x_2 \\ x_2 = x_2 \end{array}$ Setting $x_2 = -1 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = 4$: $\left(\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \begin{array}{l} x_1 = x_2 \\ x_2 = x_2 \end{array}$

Setting $x_2 = 1 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b. General Solution: $\boxed{\vec{X}(t) = c_1 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

c. Since $-2, 4$ have opposite signs, $(0,0)$ is an unstable saddle.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