

MATH 2280-001

EXAM 1

Instructions. There are 5 problems, each worth the same number of points. No calculators or electronic devices are allowed on the exam. You are allowed one side of an 8.5 by 11 inch sheet of notes.

1. Consider the ODE:

$$\frac{dy}{dx} = xy^{2/3}$$

a. Find a general solution.

b. Find a solution satisfying the initial condition. $y(0) = 0$. Show that this is not unique (find another solution).

c. Explain why the Existence-Uniqueness Theorem does not hold.

a. Separable Equation: $y^{-2/3} \frac{dy}{dx} = x$, $\int y^{-2/3} \frac{dy}{dx} dx = \int x dx$

$$\Rightarrow 3y^{1/3} = \frac{x^2}{2} + C \Rightarrow y^{1/3} = \frac{x^2}{6} + C \Rightarrow \boxed{y = \left(\frac{x^2}{6} + C\right)^3}$$

b. $y(0) = C^3 = 0 \Rightarrow C = 0$

$$\boxed{y(x) = \left(\frac{x^2}{6}\right)^3} \quad \text{or} \quad \boxed{y(x) = \frac{x^6}{6^3}}$$

By inspection, $y(x) = 0$ all x also a solution

c. O.D.E. is of form $\frac{dy}{dx} = f(x,y)$ where

$$f(x,y) = xy^{2/3}$$

f is continuous by $\frac{\partial f}{\partial y} = \frac{2}{3}xy^{-1/3}$ is not

continuous in an rectangle containing $(0,0)$.

2. Consider the autonomous differential equation

$$x' = 4x^2 - x^3 - 3x$$

- Find the equilibrium solutions.
- Determine which ones are stable, and which ones are unstable.
- Sketch the sketch the equilibrium solutions, as well as the solutions that satisfy the initial conditions $x(0) = -1$, $x(0) = 1/2$, $x(0) = 2$ and $x(0) = 4$.

(a) $x' = f(x)$ where $f(x) = 4x^2 - x^3 - 3x$.

Equilibrium solutions are solns of $f(x) = 0 \iff$

$$4x^2 - x^3 - 3x = 0 \iff -x(x^2 - 4x + 3) = 0$$

$$-x(x-1)(x-3) = 0$$

\iff $x=0, x=1, x=3$ are equilibrium solutions.

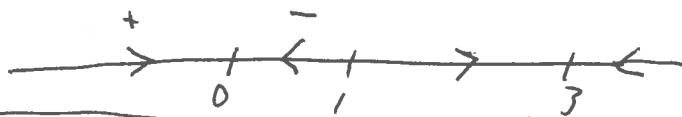
(b) Check sign of f on either sides of equilibrium solution!

$x < 0$: $f(x)$: pos. · neg. neg. = pos.

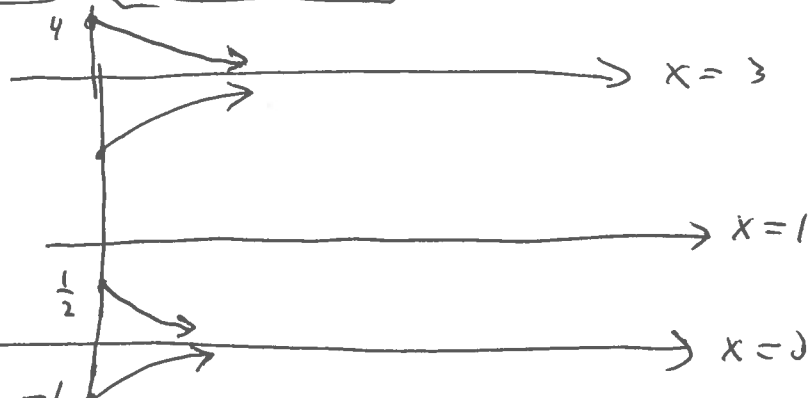
$0 < x < 1$: $f(x)$: neg. neg. neg. = neg.

$1 < x < 3$: $f(x)$: neg. pos. neg. = pos.

$3 < x$: $f(x)$: neg. pos. pos. = neg.



$x=0$ asym. stable $x=1$ unstable $x=3$ asym. stable



(c)

3. Consider a brine tank which holds 15,000 gallons of continuously mixed liquid. Let $x(t)$ be the amount of salt in the tank at time t (in hours). The inflow and outflow rates are both 150 gallons per hour, and the concentration of salt flowing in is 1 pound per 10 gallons of water.

a. From this information explain how x satisfies the differential equation

$$\frac{dx}{dt} + 0.01x = 15$$

b. If there is no salt in the tank at time $t = 0$, find $x(t)$.

c. What is the limiting amount of salt as t approaches infinity?

(a) $\frac{dx}{dt} = \text{incoming rate} - \text{outgoing rate}$

incoming rate = incoming vol. \cdot incoming concentration = $\frac{150 \text{ gal.}}{\text{hr.}} \cdot \frac{1 \text{ lb.}}{10 \text{ gal.}} = 15$

outgoing rate = outgoing vol. rate \cdot outgoing concentration = $\frac{150 \text{ gal.}}{\text{hr.}} \cdot \frac{x}{15000} = 150 \cdot \frac{x}{15000} = 0.01x$

$\Rightarrow \frac{dx}{dt} = 15 - 0.01x \Rightarrow \frac{dx}{dt} + 0.01x = 15$

(b) $x(0) = 0$: Integrating factor $p(t) = e^{\int 0.01 dt} = e^{0.01t}$

$e^{0.01t} \frac{dx}{dt} + 0.01 e^{0.01t} x = 15 e^{0.01t}$

$(e^{0.01t} x)' = 15 e^{0.01t}$ Integrate $\int (e^{0.01t} x)' dt = \int 15 e^{0.01t} dt$

$\Rightarrow e^{0.01t} x = \frac{15}{0.01} e^{0.01t} + C$. Divide by $e^{0.01t}$

$x = 1500 + C e^{-0.01t}$, $x(0) = 1500 + C = 0$

$\Rightarrow C = -1500$ $x(t) = 1500 - 1500 e^{-0.01t}$

(c) $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} 1500 - 1500 e^{-0.01t} = \boxed{1500 \text{ lb.}}$

4a. Find the general solution of the differential equation

$$y''' + 2y'' + 2y' = 0$$

b. Solve the equation

$$y'' + 2y' + 2y = 0$$

with the initial conditions $y(0) = 1$ and $y'(0) = -1$.

(a) Characteristic equation is $r^3 + 2r^2 + 2r = 0$

$$r(r^2 + 2r + 2) = 0$$

$$\Rightarrow \text{roots are } r = 0, r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm \frac{\sqrt{-4}}{2} = -1 \pm i$$

\Rightarrow Basis of solution: $y_1 = e^{0x} = 1$, $y_2 = e^{-x} \cos x$, $y_3 = e^{-x} \sin x$

general solution is $y(x) = c_1 + c_2 e^{-x} \cos x + c_3 e^{-x} \sin x$

(b) Characteristic equation is $r^2 + 2r + 2 = 0$. Already solved in (a)

$$r = -1 \pm i$$

General solution is $y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$.

$$\text{Now, } y'(x) = -c_1 e^{-x} \cos x - c_1 e^{-x} \sin x - c_2 e^{-x} \sin x + c_2 e^{-x} \cos x.$$

Plug in initial conditions!

$$y(0) = c_1 + 0 = 1 \Rightarrow c_1 = 1$$

$$y'(0) = -c_1 + 0 + 0 + c_2 = -1 \Rightarrow -1 + c_2 = -1, -1 + c_2 = -1 \Rightarrow c_2 = 0$$

$$y(x) = e^{-x} \cos x$$

5. Consider the first order linear equation

$$y' + \cos(x)y = \cos(x)$$

a. Find the general solution.

b. Find the solution with the initial condition $y(0) = 2$.

(a) Integrating factor $p(x) = e^{\int \cos x dx} = e^{\sin x}$

$$e^{\sin x} y' + \cos x e^{\sin x} y = \cos x e^{\sin x}$$

$$(e^{\sin x} y)' = \cos x e^{\sin x}$$

$$\int (e^{\sin x} y)' dx = \int \cos x e^{\sin x} dx \quad \begin{array}{l} u = \sin x \\ du = \cos x dx \end{array}$$

$$e^{\sin x} y = \int e^u du = e^u + C = e^{\sin x} + C \quad \text{Divide by } e^{\sin x} :$$

$$y = 1 + C e^{-\sin x}$$

(b) $y(0) = 1 + C = 2 \Rightarrow C = 1$

$$y(x) = 1 + e^{-\sin x}$$