Chapter 3: Linear Differential Equations and Applications

Sample Problem 9. Solving Higher Order Constant-Coefficient Equations

The Algorithm applies to constant-coefficient homogeneous linear differential equations of order N, for example equations like

$$y'' + 16y = 0, \quad y'''' + 4y'' = 0, \quad \frac{d^5y}{dx^5} + 2y''' + y'' = 0.$$

1. Find the Nth degree characteristic equation by Euler's substitution $y = e^{rx}$. For instance, $y''+16y = 0$ has characteristic equation $r^2+16 = 0$, a polynomial equation of degree $N = 2$.

2. Find all real roots and all complex conjugate pairs of roots satisfying the characteristic equation. List the N roots according to multiplicity.

3. Construct N distinct Euler solution atoms from the list of roots. Then the general solution of the differential equation is a linear combination of the Euler solution atoms with arbitrary coefficients $c_1, c_2, c_3, \ldots$.
The solution space S of the differential equation is given by

$$S = \textbf{span}(\text{the } N \text{ Euler solution atoms}).$$

Examples: Constructing Euler Solution Atoms from roots.

Three roots $0, 0, 0$ produce three atoms $e^{0x}, xe^{0x}, x^2e^{0x}$ or $1, x, x^2$.

Three roots $0, 0, 2$ produce three atoms e^{0x}, xe^{0x}, e^{2x}.

Two complex conjugate roots $2 \pm 3i$ produce two atoms $e^{2x}\cos(3x), e^{2x}\sin(3x)$.[1]

Four complex conjugate roots listed according to multiplicity as $2\pm 3i, 2\pm 3i$ produce four atoms $e^{2x}\cos(3x), e^{2x}\sin(3x),\ xe^{2x}\cos(3x), xe^{2x}\sin(3x)$.

Seven roots $1, 1, 3, 3, 3, \pm 3i$ produce seven atoms $e^x, xe^x, e^{3x}, xe^{3x}, x^2e^{3x}, \cos(3x), \sin(3x)$.

Two conjugate complex roots $a\pm bi$ $(b > 0)$ arising from roots of $(r-a)^2+b^2 = 0$ produce two atoms $e^{ax}\cos(bx), e^{ax}\sin(bx)$.

The Problem

Solve for the general solution or the particular solution satisfying initial conditions.

(a) $y'' + 16y' = 0$

(b) $y'' + 16y = 0$

(c) $y'''' + 16y'' = 0$

(d) $y'' + 16y = 0,\ y(0) = 1,\ y'(0) = -1$

(e) $y'''' + 9y'' = 0,\ y(0) = y'(0) = 0,\ y''(0) = y'''(0) = 1$

(f) The characteristic equation is $(r-2)^2(r^2-4) = 0$.

(g) The characteristic equation is $(r-1)^2(r^2-1)((r+2)^2+4) = 0$.

(h) The characteristic equation roots, listed according to multiplicity, are $0, 0, 0, -1, 2, 2, 3+4i, 3-4i$.

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[1] The Reason: $\cos(3x) = \frac{1}{2}e^{3xi} + \frac{1}{2}e^{-3xi}$ by Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. Then $e^{2x}\cos(3x) = \frac{1}{2}e^{2x+3xi} + \frac{1}{2}e^{2x-3xi}$ is a linear combination of exponentials e^{rx} where r is a root of the characteristic equation. Euler's substitution implies e^{rx} is a solution, so by superposition, so also is $e^{2x}\cos(3x)$. Similar for $e^{2x}\sin(3x)$.

Solutions to Problem 9

(a) $y''+16y'=0$ upon substitution of $y=e^{rx}$ becomes $(r^2+16r)e^{rx}=0$. Cancel e^{rx} to find the **characteristic equation** $r^2+16r=0$. It factors into $r(r+16)=0$, then the two roots r make the list $r=0,-16$. The Euler solution atoms for these roots are e^{0x}, e^{-16x}. Report the general solution $y=c_1e^{0x}+c_2e^{-16x}=c_1+c_2e^{-16x}$, where symbols c_1, c_2 stand for arbitrary constants.

(b) $y''+16y=0$ has characteristic equation $r^2+16=0$. Because a quadratic equation $(r-a)^2+b^2=0$ has roots $r=a\pm bi$, then the root list for $r^2+16=0$ is $0+4i, 0-4i$, or briefly $\pm 4i$. The Euler solution atoms are $e^{0x}\cos(4x), e^{0x}\sin(4x)$. The general solution is $y=c_1\cos(4x)+c_2\sin(4x)$, because $e^{0x}=1$.

(c) $y''''+16y''=0$ has characteristic equation $r^4+4r^2=0$ which factors into $r^2(r^2+16)=0$ having root list $0,0,0\pm 4i$. The Euler solution atoms are $e^{0x}, xe^{0x}, e^{0x}\cos(4x), e^{0x}\sin(4x)$. Then the general solution is $y=c_1+c_2x+c_3\cos(4x)+c_4\sin(4x)$.

(d) $y''+16y=0$, $y(0)=1$, $y'(0)=-1$ defines a particular solution y. The usual arbitrary constants c_1, c_2 are determined by the initial conditions. From part (b), $y=c_1\cos(4x)+c_2\sin(4x)$. Then $y'=-4c_1\sin(4x)+4c_2\cos(4x)$. Initial conditions $y(0)=1, y'(0)=-1$ imply the equations $c_1\cos(0)+c_2\sin(0)=1, -4c_1\sin(0)+4c_2\cos(0)=-1$. Using $\cos(0)=1$ and $\sin(0)=0$ simplifies the equations to $c_1=1$ and $4c_2=-1$. Then the particular solution is $y=c_1\cos(4x)+c_2\sin(4x)=\cos(4x)-\frac{1}{4}\sin(4x)$.

(e) $y''''+9y''=0$, $y(0)=y'(0)=0$, $y''(0)=y'''(0)=1$ is solved like part (d). First, the characteristic equation $r^4+9r^2=0$ is factored into $r^2(r^2+9)=0$ to find the root list $0,0,0\pm 3i$. The Euler solution atoms are $e^{0x}, xe^{0x}, e^{0x}\cos(3x), e^{0x}\sin(3x)$, which implies the general solution $y=c_1+c_2x+c_3\cos(3x)+c_4\sin(3x)$. We have to find the derivatives of y: $y'=c_2-3c_3\sin(3x)+3c_4\cos(3x)$, $y''=-9c_3\cos(3x)-9c_4\sin(3x)$, $y'''=27c_3\sin(3x)-27c_4\cos(3x)$. The initial conditions give four equations in four unknowns c_1, c_2, c_3, c_4:

$$\begin{array}{ccccccccc} c_1 & + & c_2(0) & + & c_3\cos(0) & + & c_4\sin(0) & = & 0, \\ & & c_2 & - & 3c_3\sin(0) & + & 3c_4\cos(0) & = & 0, \\ & & & - & 9c_3\cos(0) & - & 9c_4\sin(0) & = & 1, \\ & & & & 27c_3\sin(0) & - & 27c_4\cos(0) & = & 1, \end{array}$$

which has invertible coefficient matrix $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -9 & 0 \\ 0 & 0 & 0 & -27 \end{pmatrix}$ and right side vector $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$. The solution is $c_1=c_2=1/9$, $c_3=-1/9, c_4=-1/27$. Then the particular solution is $y=c_1+c_2x+c_3\cos(3x)+c_4\sin(3x)=\frac{1}{9}+\frac{1}{9}x-\frac{1}{9}\cos(3x)-\frac{1}{27}\sin(3x)$

(f) The characteristic equation is $(r-2)^2(r^2-4)=0$. Then $(r-2)^3(r+2)=0$ with root list $2,2,2,-2$ and Euler atoms $e^{2x}, xe^{2x}, x^2e^{2x}, e^{-2x}$. The general solution is a linear combination of these four atoms.

(g) The characteristic equation is $(r-1)^2(r^2-1)((r+2)^2+4)=0$. The root list is $1,1,1,-1,-2\pm 2i$ with Euler atoms $e^x, xe^x, x^2e^x, e^{-x}, e^{-2x}\cos(2x), e^{-2x}\sin(2x)$. The general solution is a linear combination of these six atoms.

(h) The characteristic equation roots, listed according to multiplicity, are $0,0,0,-1,2,2,3+4i,3-4i$. Then the Euler solution atoms are $e^{0x}, xe^{0x}, x^2e^{0x}, e^{-x}, e^{2x}, xe^{2x}, e^{3x}\cos(4x), e^{3x}\sin(4x)$. The general solution is a linear combination of these eight atoms.

Chapter 7: Laplace Theory

Sample Problem 10.

> Laplace theory implements the *method of quadrature* for higher order differential equations, linear systems of differential equations, and certain partial differential equations.
>
> Laplace's method **solves differential equations**.

The Problem. Solve by table methods or Laplace's method.

(a) Forward table. Find $\mathcal{L}(f(t))$ for $f(t) = te^{2t} + 2t\sin(3t) + 3e^{-t}\cos(4t)$.

(b) Backward table. Find $f(t)$ for

$$\mathcal{L}(f(t)) = \frac{16}{s^2+4} + \frac{s+1}{s^2-2s+10} + \frac{2}{s^2+16}.$$

(c) Solve the initial value problem $x''(t) + 256x(t) = 1$, $x(0) = 1$, $x'(0) = 0$.

Solution (a).

$$\begin{array}{rcll}
\mathcal{L}(f(t)) & = & \mathcal{L}(te^{2t} + 2t\sin(3t) + 3e^{-t}\cos(4t)) & \\
& = & \mathcal{L}(te^{2t}) + 2\mathcal{L}(t\sin(3t)) + 3\mathcal{L}(e^{-t}\cos(4t)) & \text{Linearity} \\
& = & -\frac{d}{ds}\mathcal{L}(e^{2t}) - 2\frac{d}{ds}\mathcal{L}(\sin(3t)) + 3\mathcal{L}(e^{-t}\cos(4t)) & \text{Differentiation rule} \\
& = & -\frac{d}{ds}\mathcal{L}(e^{2t}) - 2\frac{d}{ds}\mathcal{L}(\sin(3t)) + 3\left.\mathcal{L}(\cos(4t))\right|_{s=s+1} & \text{Shift rule} \\
& = & -\frac{d}{ds}\frac{1}{s-2} - 2\frac{d}{ds}\frac{3}{s^2+9} + 3\left.\frac{s}{s^2+16}\right|_{s=s+1} & \text{Forward table} \\
& = & \frac{1}{(s-2)^2} + \frac{12s}{(s^2+9)^2} + 3\frac{s+1}{(s+1)^2+16} & \text{Calculus}
\end{array}$$

Solution (b).

$$\begin{array}{rcll}
\mathcal{L}(f(t)) & = & \frac{16}{s^2+4} + \frac{s+1}{s^2-2s+10} + \frac{2}{s^2+16} & \\
& = & 8\frac{2}{s^2+4} + \frac{s+1}{(s-1)^2+9} + \frac{1}{2}\frac{4}{s^2+16} & \text{Prep for backward table} \\
& = & 8\mathcal{L}(\sin 2t) + \frac{s+1}{(s-1)^2+9} + \frac{1}{2}\mathcal{L}(\sin 4t) & \text{backward table} \\
& = & 8\mathcal{L}(\sin 2t) + \left.\frac{s+2}{s^2+9}\right|_{s=s-1} + \frac{1}{2}\mathcal{L}(\sin 4t) & \text{shift rule} \\
& = & 8\mathcal{L}(\sin 2t) + \left.\mathcal{L}(\cos 3t + \frac{2}{3}\sin 3t)\right|_{s=s-1} + \frac{1}{2}\mathcal{L}(\sin 4t) & \text{backward table} \\
& = & 8\mathcal{L}(\sin 2t) + \mathcal{L}(e^t\cos 3t + e^t\frac{2}{3}\sin 3t) + \frac{1}{2}\mathcal{L}(\sin 4t) & \text{shift rule} \\
& = & \mathcal{L}(8\sin 2t) + e^t\cos 3t + e^t\frac{2}{3}\sin 3t + \frac{1}{2}\sin 4t) & \text{Linearity} \\
f(t) & = & 8\sin 2t + e^t\cos 3t + e^t\frac{2}{3}\sin 3t + \frac{1}{2}\sin 4t & \text{Lerch's cancel rule}
\end{array}$$

Solution (c).

$$\begin{array}{lcll}
\mathcal{L}(x''(t) + 256x(t)) & = & \mathcal{L}(1) & \mathcal{L}\text{ acts like matrix mult} \\
s\mathcal{L}(x') - x'(0) + 256\mathcal{L}(x) & = & \mathcal{L}(1) & \text{Parts rule} \\
s(s\mathcal{L}(x) - x(0)) - x'(0) + 256\mathcal{L}(x) & = & \mathcal{L}(1) & \text{Parts rule} \\
s^2\mathcal{L}(x) - s + 256\mathcal{L}(x) & = & \mathcal{L}(1) & \text{Use } x(0) = 1, x'(0) = 0 \\
(s^2 + 256)\mathcal{L}(x) & = & s + \mathcal{L}(1) & \text{Collect } \mathcal{L}(x) \text{ left}
\end{array}$$

$$\begin{array}{rcll}
\mathcal{L}(x) & = & \frac{s+\mathcal{L}(1)}{(s^2+256)} & \text{Isolate } \mathcal{L}(x) \text{ left} \\
\mathcal{L}(x) & = & \frac{s+1/s}{(s^2+256)} & \text{Forward table} \\
\mathcal{L}(x) & = & \frac{s^2+1}{s(s^2+256)} & \text{Algebra} \\
\mathcal{L}(x) & = & \frac{A}{s} + \frac{Bs+C}{s^2+256} & \text{Partial fractions} \\
\mathcal{L}(x) & = & A\mathcal{L}(1) + B\mathcal{L}(\cos 16t) + \frac{C}{16}\mathcal{L}(\sin 16t) & \text{Backward table} \\
\mathcal{L}(x) & = & \mathcal{L}(A + B\cos 16t + \frac{C}{16}\sin 16t) & \text{Linearity} \\
x(t) & = & A + B\cos 16t + \frac{C}{16}\sin 16t & \text{Lerch's rule}
\end{array}$$

The partial fraction problem remains:

$$\frac{s^2+1}{s(s^2+256)} = \frac{A}{s} + \frac{Bs+C}{s^2+256}$$

This problem is solved by clearing the fractions, then swapping sides of the equation, to obtain

$$A(s^2+256) + (Bs+C)(s) = s^2+1.$$

Substitute three values for s to find 3 equations in 3 unknowns A, B, C:

$$\begin{array}{lll} s=0 & 256A & = 1 \\ s=1 & 257A+B+C & = 2 \\ s=-1 & 257A+B-C & = 2 \end{array}$$

Then $A = 1/256, B = 255/256, C = 0$ and finally

$$x(t) = A + B\cos 16t + \frac{C}{16}\sin 16t = \frac{1+255\cos 16t}{256}$$

Answer Checks

```
# Sample Problem 10
# answer check part (a)
f:=t*exp(2*t)+2*t*sin(3*t)+3*exp(-t)*cos(4*t);
with(inttrans): # load laplace package
laplace(f,t,s);
# The last two fractions simplify to 3(s+1)/((s+1)^2+16).
# answer check part (b)
F:=16/(s^2+4)+(s+1)/(s^2-2*s+10)+2/(s^2+16);
invlaplace(F,s,t);
# answer check part (c)
de:=diff(x(t),t,t)+256*x(t)=1;ic:=x(0)=1,D(x)(0)=0;
dsolve([de,ic],x(t));
# answer check part (c), partial fractions
convert((s^2+1)/(s*(s^2+256)),parfrac,s);
```

The output appears on the next page

```
> # Sample Problem 10
> # answer check part (a)
> f:=t*exp(2*t)+2*t*sin(3*t)+3*exp(-t)*cos(4*t);
```

$$f := t\,e^{2t} + 2\,t\sin(3\,t) + 3\,e^{-t}\cos(4\,t) \tag{1}$$

```
> with(inttrans): # load laplace package
> laplace(f,t,s);
```

$$\frac{1}{(s-2)^2} + \frac{12\,s}{\left(s^2+9\right)^2} + \frac{3}{2\,(s+1-4\,\mathrm{I})} + \frac{3}{2\,(s+1+4\,\mathrm{I})} \tag{2}$$

```
> # The last two fractions simplify to 3(s+1)/((s+1)^2+16).
> # answer check part (b)
> F:=16/(s^2+4)+(s+1)/(s^2-2*s+10)+2/(s^2+16);
```

$$F := \frac{16}{s^2+4} + \frac{s+1}{s^2-2\,s+10} + \frac{2}{s^2+16} \tag{3}$$

```
> invlaplace(F,s,t);
```

$$8\sin(2\,t) + \frac{1}{2}\sin(4\,t) + \frac{1}{3}\,e^{t}\,(3\cos(3\,t) + 2\sin(3\,t)) \tag{4}$$

```
> # answer check part (c)
> de:=diff(x(t),t,t)+256*x(t)=1;ic:=x(0)=1,D(x)(0)=0;
```

$$de := \frac{d^2}{dt^2}\,x(t) + 256\,x(t) = 1$$

$$ic := x(0) = 1,\ D(x)(0) = 0 \tag{5}$$

```
> dsolve([de,ic],x(t));
```

$$x(t) = \frac{1}{256} + \frac{255}{256}\cos(16\,t) \tag{6}$$

```
> # answer check part (c), partial fractions
> convert((s^2+1)/(s*(s^2+256)),parfrac,s);
```

$$\frac{255}{256}\,\frac{s}{s^2+256} + \frac{1}{256\,s} \tag{7}$$