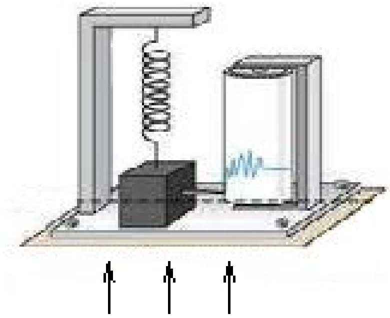
Chapter 3. Sample Problem 5. Vertical Motion Seismoscope

The 1875 **horizontal motion seismoscope** of F. Cecchi (1822-1887) reacted to an earthquake. It started a clock, and then it started motion of a recording surface, which ran at a speed of 1 cm per second for 20 seconds. The clock provided the observer with the earthquake hit time.



A Simplistic Vertical Motion Seismoscope

The apparently stationary heavy mass on a spring writes with the attached stylus onto a rotating drum, as the ground moves up. The generated trace is $x(t)$.

The motion of the heavy mass m in the figure can be modeled initially by a forced spring-mass system with damping. The initial model has the form

$$mx'' + cx' + kx = f(t)$$

where $f(t)$ is the vertical ground force due to the earthquake. In terms of the vertical ground motion $u(t)$, we write via Newton's second law the force equation $f(t) = -mu''(t)$ (compare to falling body $-mg$). The final model for the motion of the mass is then

$$\begin{cases} x''(t) + 2\beta\Omega_0 x'(t) + \Omega_0^2 x(t) = -u''(t), \\ \dfrac{c}{m} = 2\beta\Omega_0, \quad \dfrac{k}{m} = \Omega_0^2, \\ x(t) = \text{center of mass position measured from equilibrium}, \\ u(t) = \text{vertical ground motion due to the earthquake}. \end{cases} \tag{1}$$

Terms **seismoscope, seismograph, seismometer** refer to the device in the figure. Some observations:

Slow ground movement means $x' \approx 0$ and $x'' \approx 0$, then (1) implies $\Omega_0^2 x(t) = -u''(t)$. The seismometer records ground acceleration.

Fast ground movement means $x \approx 0$ and $x' \approx 0$, then (1) implies $x''(t) = -u''(t)$. The seismometer records ground displacement.

A **release test** begins by starting a vibration with u identically zero. Two successive maxima $(t_1, x_1), (t_2, x_2)$ are recorded. This experiment determines constants β, Ω_0.

The objective of (1) is to determine $u(t)$, by knowing $x(t)$ from the seismograph.

The Problem.

(a) Explain how a **release test** can find values for β, Ω_0 in the model $x''+2\beta\Omega_0 x'+\Omega_0^2 x = 0$.

(b) Assume the seismograph trace can be modeled at time $t = 0$ (a time after the earthquake struck) by $x(t) = Ce^{-at}\sin(bt)$ for some positive constants C, a, b. Assume a release test determined $2\beta\Omega_0 = 12$ and $\Omega_0^2 = 100$. Explain how to find a formula for the ground motion $u(t)$, then provide a formula for $u(t)$, using technology.

Solution to the Seismoscope Problem.

(a) A **release test** is an experiment which provides initial data $x(0) > 0$, $x'(0) = 0$ to the seismoscope mass. The model is $x'' + 2\beta\Omega_0 x' + \Omega_0^2 x = 0$ (ground motion zero). The recorder graphs $x(t)$ during the experiment, until two successive maxima $(t_1, x_1), (t_2, x_2)$ appear in the graph. This is enough information to find values for β, Ω_0.

In an under-damped oscillation, the characteristic equation is $(r+p)^2 + \omega^2 = 0$ corresponding to complex conjugate roots $-p \pm \omega i$. The phase-amplitude form is $x(t) = Ce^{-pt}\cos(\omega t - \alpha)$, with period $2\pi/\omega$.

The equation $x'' + 2\beta\Omega_0 x' + \Omega_0^2 x = 0$ has characteristic equation $(r+\beta)^2 + \Omega_0^2 = 0$. Therefore $x(t) = Ce^{-\beta t}\cos(\Omega_0 t - \alpha)$.

The period is $t_2 - t_1 = 2\pi/\Omega_0$. Therefore, Ω_0 is known. The maxima occur when the cosine factor is 1, therefore

$$\frac{x_2}{x_1} = \frac{Ce^{-\beta t_2}\cdot 1}{Ce^{-\beta t_1}\cdot 1} = e^{-\beta(t_2 - t_1)}.$$

This equation determines β.

(b) The equation $-u''(t) = x''(t) + 2\beta\Omega_0 x'(t) + \Omega_0^2 x(t)$ (the model written backwards) determines $u(t)$ in terms of $x(t)$. If $x(t)$ is known, then this is a quadrature equation for the ground motion $u(t)$.

For the example $x(t) = Ce^{-at}\sin(bt)$, $2\beta\Omega_0 = 12, \Omega_0^2 = 100$, then the quadrature equation is

$$-u''(t) = x''(t) + 12x'(t) + 100x(t).$$

After substitution of $x(t)$, the equation becomes

$$-u''(t) = C\mathrm{e}^{-at}\left(\sin(bt)\,a^2 - \sin(bt)\,b^2 - 2\cos(bt)\,ab - 12\sin(bt)\,a + 12\cos(bt)\,b + 100\sin(bt)\right)$$

which can be integrated twice using Maple, for simplicity. All integration constants will be assumed zero. The answer:

$$\begin{aligned} u(t) &= \frac{C\mathrm{e}^{-at}\left(12\,a^2 b + 12\,b^3 - 200\,ab\right)\cos(bt)}{\left(a^2+b^2\right)^2} \\ &\quad - \frac{C\mathrm{e}^{-at}\left(a^4 + 2\,a^2 b^2 + b^4 - 12\,a^3 - 12\,ab^2 + 100\,a^2 - 100\,b^2\right)\sin(bt)}{\left(a^2+b^2\right)^2} \end{aligned}$$

The Maple session has this brief input:

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de:=-diff(u(t),t,t) = diff(x(t),t,t) + 12*diff(x(t),t) + 100* x(t);
x:=t->C*exp(-a*t)*sin(b*t);
dsolve(de,u(t));subs(_C1=0,_C2=0,%);
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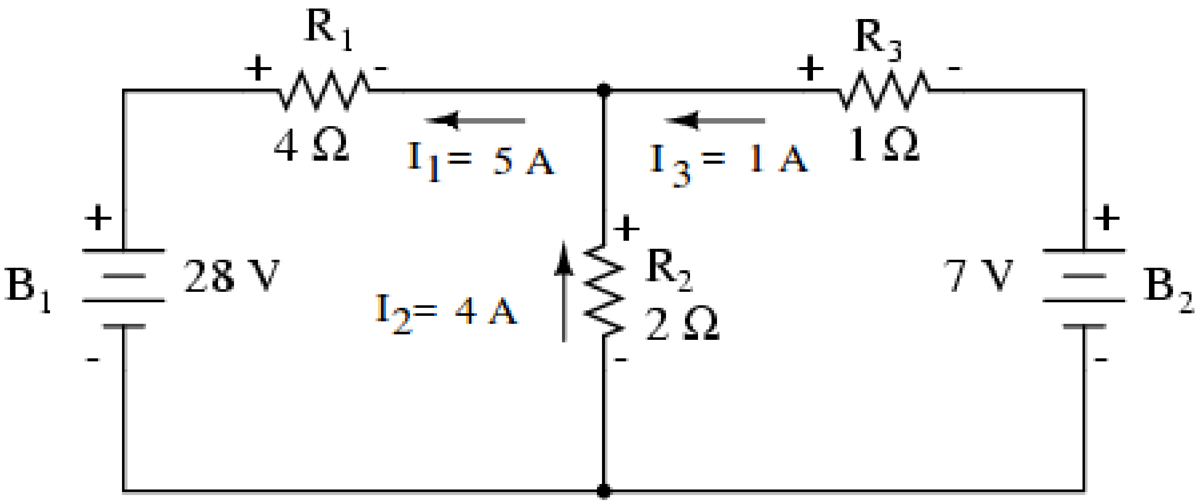
Chapter 3. Sample Problem 6. Resistive Network with 2 Loops and DC Sources.



The **Branch Current Method** can be used to find a 3×3 linear system for the **branch currents** I_1, I_2, I_3.

$$\begin{array}{rcrcrcrl} I_1 & - & I_2 & - & I_3 & = & 0 & \text{KCL, upper node} \\ 4I_1 & + & 2I_2 & & & = & 28 & \text{KVL, left loop} \\ & & 2I_2 & - & I_3 & = & 7 & \text{KVL, right loop} \end{array}$$

Symbol **KCL** means *Kirchhoff's Current Law*, which says the algebraic sum of the currents at a node is zero. Symbol **KVL** means *Kirchhoff's Voltage Law*, which says the algebraic sum of the voltage drops around a closed loop is zero.

(a) Solve the equations to verify the currents reported in the figure: $I_1 = 5, I_2 = 4, I_3 = 1$ Amperes.

(b) Compute the voltage drops across resistors R_1, R_2, R_3. Answer: $20, 8, 1$ volts.

References. Edwards-Penney 3.7, electric circuits. All About Circuits Volume I – DC, by T. Kuphaldt:
http://www.allaboutcircuits.com/.
Course slides on Electric Circuits:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/electricalCircuits.pdf`
Solved examples of electrical networks can be found in the lecture notes of Ruye Wang:
http://fourier.eng.hmc.edu/e84/lectures/ch2/node2.html.

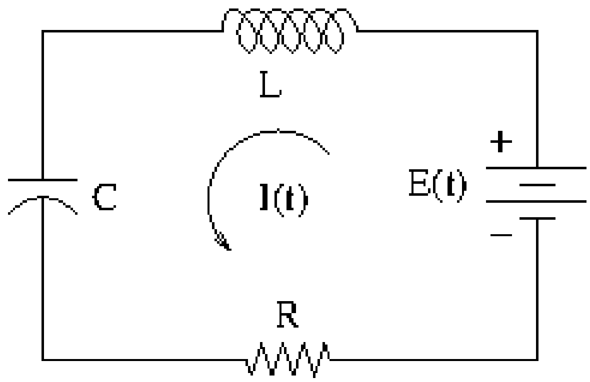
Solutions to Problem 6

Part (a). Write the system as a matrix equation $A\vec{x} = \vec{b}$. Solve for $\vec{x}$ by any method, including technology, to get $\vec{x} = \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix}$, whose components are currents I_1, I_2, I_3.

Part (b). Use Ohm's Law $V = RI$ to compute V, which is the voltage drop across resistor R. Use the answers from part (a).

Handwritten solutions to Problem 6 appear below, after the solution to Problem 7.

Chapter 3. Sample Problem 7. *RLC*-**Circuit**



The Problem. Suppose $E = 100\sin(20t)$, $L = 5$ H, $R = 250\,\Omega$ and $C = 0.002$ F. The model for the charge $Q(t)$ is $LQ'' + RQ' + \frac{1}{C}Q = E(t)$.

(a) Differentiate the charge model and substitute $I = \frac{dQ}{dt}$ to obtain the current model $5I'' + 250I' + 500I = 2000\cos(20t)$.

(b) Find the **reactance** $S = \omega L - \frac{1}{\omega C}$, where $\omega = 20$ is the input frequency, the natural frequency of $E = 100\sin(20t)$ and $E' = 2000\cos(20t)$.

(c) Substitute $I = A\cos(20t) + B\sin(20t)$ into the current model (a) and solve for $A = \frac{-12}{109}$, $B = \frac{40}{109}$. Then the steady-state current is

$$I(t) = A\cos(20t) + B\sin(20t) = \frac{-12\cos(20t) + 40\sin(20t)}{109}.$$

(d) Write the answer in (c) in phase-amplitude form $I = I_0\sin(20t - \delta)$ with $I_0 > 0$ and $\delta \geq 0$. Then compute the **time lag** δ/ω.

Answers: $I_0 = \frac{4}{\sqrt{109}}$, $\delta = \arctan(3/10)$, $\delta/\omega = 0.01457$.

References

Course slides on Electric Circuits:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/electricalCircuits.pdf`
Edwards-Penney *Differential Equations and Boundary Value Problems*, sections 3.4, 3.5, 3.6, 3.7.

Solutions to Problem 7

Part (a) Start with $5Q'' + 250Q' + 500Q = 100\sin(20t)$. Differentiate across to get $5Q''' + 250Q'' + 500Q' = 2000\cos(20t)$. Change Q' to I.

Part (b) $S = (20)(5) - 1/(20 * 0.002) = 75$

Part (c) It helps to use the differential equation $u'' + 400u = 0$ satisfied by both $u_1 = \cos(20t)$ and $u_2 = \sin(20t)$. Functions u_1, u_2 are Euler solution atoms, hence independent. Along the solution path, we'll use $u_1' = -20\sin(20t) = -20u_2$ and $u_2' = 20\cos(20t) = 20u_1$. The arithmetic is simplified by dividing the equation first by 5. We then substitute $I = Au_1 + Bu_2$.

$$
\begin{aligned}
&I'' + 50I' + 100I = 400\sin(20t)\\
&A(u_1'' + 50u_1' + 100u_1) + B(u_2'' + 50u_2' + 100u_2) = 400\sin(20t)\\
&A(-400u_1 + 50(-20u_2) + 100u_1) + B(-400u_2 + 50(20u_1) + 100u_2) = 400\sin(20t)\\
&(-400A + 100A + 1000B)u_1 + (-1000A - 400B + 100B)u_2 = 400u_2
\end{aligned}
$$

By independence of u_1, u_2, coefficients of u_1, u_2 on each side of the equation must match. The linear algebra property is called *unique representation of linear combinations*. This implies the 2×2 system of equations

$$
\begin{array}{rcrcl}
-300A & + & 1000B & = & 0,\\
-1000A & - & 300B & = & 400.
\end{array}
$$

The solution by Cramer's rule (the easiest method) is $A = -12/109, B = 40/109$. Then the steady-state current is

$$I(t) = A\cos(20t) + B\sin(20t) = \frac{-12\cos(20t) + 40\sin(20t)}{109}.$$

The **steady-state current** is defined to be the sum of those terms in the general solution of the differential equation that remain after all terms that limit to zero at $t = \infty$ have been removed. The logic is that only these terms contribute to a graphic or to a numerical calculation after enough time has passed (as $t \to \infty$).

Part (d) Let $\cos(\delta) = B/I_0, \sin(\delta) = -A/I_0, I_0 = \sqrt{A^2 + B^2}$. Use the trig identity

$$\sin(a - b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$

to rearrange the current formula as follows:

$$I(t) = A\cos(20t) + B\sin(20t) = I_0(\sin(20t)\cos(\delta) - \sin(\delta)\cos(20t)) = I_0\sin(20t - \delta).$$

Compute $I_0 = \sqrt{A^2 + B^2} = \frac{4}{\sqrt{109}}$. Compute $\tan(\delta) = \frac{\sin\delta}{\cos\delta} = -A/B = 12/40$. Then $\delta = \arctan(12/40)$ and finally $\delta/\omega = \arctan(3/10)/20 = 0.01457$.

Another method, using Edwards-Penney Section 3.7: Compute the **impedance** $Z = \sqrt{R^2 + S^2} = \sqrt{250^2 + 75^2} = \sqrt{68125} = 25\sqrt{109}$ and then $I_0 = E_0/Z = 4/\sqrt{109}$. The phase $\delta = \arctan(S/R) = \arctan(75/250) = \arctan(3/10)$. Then the time lag is $\delta/\omega = \frac{\arctan(0.3)}{20} = 0.01457$.