
Chapter 3. Sample Problem 1. Harmonic Vibration

A mass of $m = 250$ grams attached to a spring of Hooke's constant k undergoes free undamped vibration. At equilibrium, the spring is stretched 25 cm by a force of 8 Newtons. At time $t = 0$, the spring is stretched 0.5 m and the mass is set in motion with initial velocity 5 m/s directed away from equilibrium. Find:

(a) The numerical value of Hooke's constant k.

(b) The initial value problem for vibration $x(t)$.

Solution

(a): Hooke's law Force=k(elongation) is applied with force 8 Newtons and elongation $25/100 = 1/4$ meter. Equation $8 = k(1/4)$ implies $k = 32$ N/m.

(b): Given $m = 250/1000$ kg and $k = 32$ N/m from part (a), then the free vibration model $mx'' + kx = 0$ becomes $\frac{1}{4}x'' + 32x = 0$. Initial conditions are $x(0) = 0.5$ m and $x'(0) = 5$ m/s. The initial value problem is

$$\begin{cases} \dfrac{d^2x}{dt^2} + 128x = 0, \\ x(0) = 0.5, \\ x'(0) = 5. \end{cases}$$

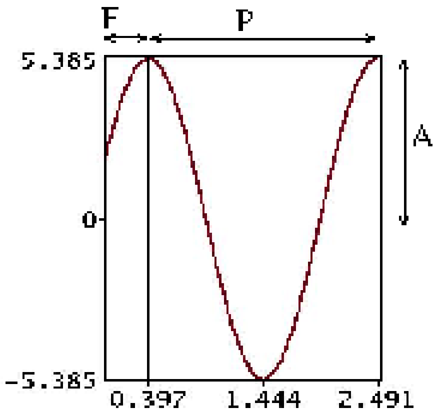
Chapter 3. Sample Problem 2. Phase-Amplitude Conversion
Write the vibration equation

$$x(t) = 2\cos(3t) + 5\sin(3t)$$

in phase-amplitude form $x = A\cos(\omega t - \alpha)$. Create a graphic of $x(t)$ with labels for period, amplitude and phase shift.

Solution
The answer and the graphic appear below.

$$x(t) = \sqrt{29}\cos(3t - 1.190289950) = \sqrt{29}\cos(3(t - 0.3967633167)).$$



Harmonic Oscillation

The graph of $2\cos(3t) + 5\sin(3t)$. It has amplitude $A = \sqrt{29} = 5.385$, period $P = 2\pi/3$ and phase shift $F = 0.3967633167$. The graph is on $0 \le t \le P + F$.

Algebra Details. The plan is to re-write $x(t)$ in the form $x(t) = A\cos(\omega t - \alpha)$, called the phase-amplitude form of the harmonic oscillation.

Start with $x(t) = 2\cos(3t) + 5\sin(3t)$. Trig identity $x(t) = A\cos(\omega t - \alpha) = A\cos(\alpha)\cos(\omega t) + A\sin(\alpha)\sin(\omega t)$ causes the definitions

$$\omega = 3, \quad A\cos(\alpha) = 2, \quad A\sin(\alpha) = 5.$$

The Pythagorean identity $\cos^2\alpha + \sin^2\alpha = 1$ implies $A^2 = 2^2 + 5^2 = 29$ and then the amplitude is $A = \sqrt{29}$. Because $\cos\alpha = 2/A$, $\sin\alpha = 5/A$, then both the sine and cosine are positive, placing angle α in quadrant I. Divide equations $\cos\alpha = 2/A$, $\sin\alpha = 5/A$ to obtain $\tan(\alpha) = 5/2$, which by calculator implies $\alpha = \arctan(5/2) = 1.190289950$ radians or 68.19859051 degrees. Then $x(t) = A\cos(\omega t - \alpha) = \sqrt{29}\cos(3t - 1.190289950)$.

Computer Details. Either equation for $x(t)$ can be used to produce a computer graphic. A hand-drawn graphic would use only the phase-amplitude form. The period is $P = 2\pi/\omega = 2\pi/3$. The amplitude is $A = \sqrt{29} = 5.385164807$ and the phase shift is $F = \alpha/\omega = 0.3967633167$. The graph is on $0 \le t \le P + F$.

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# Maple
F:=evalf(arctan(5/2)/3); P:=2*Pi/3;A:=sqrt(29);
X:=t->2*cos(3*t)+5*sin(3*t);
opts:=xtickmarks=[0,F,P/2+F,P+F],ytickmarks=[-A,0,A],
      axes=boxed,thickness=3,labels=["",""];
plot(X(t),t=0..P+F,opts);
```

Chapter 3. Sample Problem 3. Undamped Spring-Mass System
A mass of 6 Kg is attached to a spring that elongates 20 centimeters due to a force of 12 Newtons. The motion starts at equilibrium with velocity -5 m/s. Find an equation for $x(t)$ using the free undamped vibration model $mx'' + kx = 0$.

Solution

The answer is $x(t) = -\sqrt{\frac{5}{2}}\sin(\sqrt{10}t)$.

The mass is $m = 6$ kg. Hooke's law $F = kx$ is applied with $F = 12$ N and $x = 20/100$ m. Then Hooke's constant is $k = 60$ N/m. Initial conditions are $x(0) = 0$ m (equilibrium) and $x'(0) = -5$ m/s.

The Model.

$$\begin{cases} 6\dfrac{d^2x}{dt^2} + 60x = 0, \\ x(0) = 0, \\ x'(0) = -5. \end{cases}$$

Solving the Model.

The characteristic equation $6r^2 + 60 = 0$ is solved for $r = \pm i\sqrt{10}$, then the Euler solution atoms are $\cos(\sqrt{10}t)$, $\sin(\sqrt{10}t)$ and we write the general solution as

$$x(t) = c_1\cos(\sqrt{10}t) + c_2\sin(\sqrt{10}t).$$

The task remaining is determination of constants c_1, c_2 subject to initial conditions $x(0) = 0$, $x'(0) = -5$. The linear algebra problem uses the derivative formula

$$x'(t) = -\sqrt{10}c_1\sin(\sqrt{10}t) + \sqrt{10}c_2\cos(\sqrt{10}t).$$

The 2×2 system of linear algebraic equations for c_1, c_2 is obtained from the two equations $x(0) = 0$, $x'(0) = -5$ as follows.

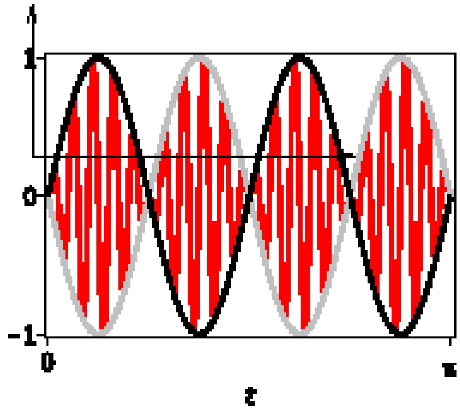
$$\left\{\begin{array}{rcrcrl} \cos(0)c_1 & + & \sin(0)c_2 & = & 0, & \text{Equation } x(0) = 0 \\ -\sqrt{10}\sin(0)c_1 & + & \sqrt{10}\cos(0)c_2 & = & -5, & \text{Equation } x'(0) = -5 \end{array}\right.$$

Because $\cos(0) = 1$, $\sin(0) = 0$, then $c_1 = 0$ and $c_2 = -5/\sqrt{10} = -\sqrt{5/2}$. Insert answers c_1, c_2 into the general solution to find the answer to the initial value problem

$$x(t) = -\sqrt{\frac{5}{2}}\sin(\sqrt{10}t).$$

Chapter 3. Sample Problem 4. Beats

The physical phenomenon of **beats** refers to the periodic interference of two sound waves of slightly different frequencies. A destructive interference occurs during a very brief interval, so our impression is that the sound periodically stops, only briefly, and then starts again with a beat, a section of sound that is instantaneously loud again. Human heartbeat uses the same terminology. Our pulse rate is $40 - 100$ **beats** per minute at rest. An illustration of the graphical meaning appears in the figure below.



Beats

Shown in red is a periodic oscillation $x(t) = 2\sin 4t \sin 40t$ with rapidly–varying factor $\sin 40t$ and the two slowly–varying envelope curves $x_1(t) = 2\sin 4t$ (black), $x_2(t) = -2\sin 4t$ (grey).

The undamped, forced spring-mass problem $x'' + 1296x = 640\cos(44t)$, $x(0) = x'(0) = 0$ has by trig identities the solution $x(t) = \cos(36t) - \cos(44t) = 2\sin 4t \sin 40t$.

A key example is piano tuning. A tuning fork is struck, then the piano string is tuned until the beats are not heard. The number of beats per second (unit Hz) is approximately the frequency difference between the two sources, e.g., two tuning forks of frequencies 440 Hz and 437 Hz would produce 3 beats per second.

The average human ear can detect beats only if the two interfering sound waves have a frequency difference of about 7 Hz or less. Ear-tuned pianos are subject to the same human ear limitations. Two piano keys are more than 7 Hz apart, even for a badly tuned piano, which is why simultaneously struck piano keys are heard as just one sound (no beats).

The beat we hear corresponds to maxima in the figure. We see not the two individual sound waves, but their **superposition**. When the tuning fork and the piano string have the same exact frequency ω, then the figure would show a simple harmonic wave, because the two sounds would **superimpose** to a graph that looks like $\cos(\omega t - \alpha)$.

The origin of the phenomenon of **beats** can be seen from the formula

$$x(t) = 2\sin at \sin bt.$$

There is no sound when $x(t) \approx 0$: this is when destructive interference occurs. When a is small compared to b, e.g., $a = 4$ and $b = 40$, then there are long intervals between the zeros of $A(t) = 2\sin at$, at which destructive interference occurs. Otherwise, the amplitude of the sound wave is the average value of $A(t)$, which is 1. The sound stops at a zero of $A(t)$ and then it is rapidly loud again, causing the beat.

The Problem. Solve the initial value problem

$$x'' + 1296x = 640\cos(44t), \quad x(0) = x'(0) = 0$$

by undetermined coefficients and linear algebra, obtaining the solution $x(t) = \cos(36t) - \cos(44t)$. Then show the trig details for $x(t) = 2\sin(4t)\sin(40t)$. Finally, graph $x(t)$ and its slowly varying envelope curves on $0 \le t \le \pi$.

Solution to Problem 4.

The trial solution for undetermined coefficients has the form $x = d_1\cos(44t) + d_2\sin(44t)$. This is the Rule I trial solution. But Rule II does not modify the trial solution, because the Euler solution atoms $\cos(44t), \sin(44t)$ are not solutions of the homogeneous equation $x'' + 1296x = 0$.

Insert the trial solution into $x'' + 1296x = 640\cos(44t)$ to obtain the equation

$$(1296 - 44^2)d_1\cos(44t) + (1296 - 44^2)d_2\sin(44t) = 640\cos(44t).$$

Then matching atoms across the equal sign implies $d_1 = 640/(1296 - 44^2) = -1$, $d_2 = 0$. The particular solution is the trial solution with $d_1 = -1$, $d_2 = 0$. The formula obtained so far is

$$x_p(t) = -\cos(44t).$$

The homogeneous solution $x_h(t)$ is found from the characteristic equation $r^2 + 1296 = 0$, with complex conjugate roots $r = \pm 36i$. Then

$$x_h(t) = c_1\cos(36t) + c_2\sin(36t).$$

The initial conditions $x(0) = x'(0) = 0$ are used together with the general solution and its derivative

$$\begin{aligned} x(t) &= x_h(t) + x_p(t) &&= c_1\cos(36t) + c_2\sin(36t) - \cos(44t) \\ x'(t) &= x_h'(t) + x_p'(t) &&= -36c_1\sin(36t) + 36c_2\cos(36t) + 44\sin(44t) \end{aligned}$$

to obtain the 2×2 linear algebraic system of equations

$$\begin{cases} \cos(0)c_1 + \sin(0)c_2 = \cos(0), & \text{Equation } x(0) = 0 \\ -36\sin(0)c_1 + 36\cos(0)c_2 = -44\sin(0). & \text{Equation } x'(0) = 0 \end{cases}$$

Then $c_1 = 1, c_2 = 0$ and $x(t) = \cos(36t) - \cos(44t)$.

The trig identities $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$, $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ are subtracted to give $\cos(a-b) - \cos(a+b) = 2\sin(a)\sin(b)$. Let $a - b = 36t, a + b = 44t$ and solve for $a = 40t, b = 4t$. Then

$$\begin{aligned} x(t) &= \cos(36t) - \cos(44t) \\ &= \cos(a-b) - \cos(a+b) \\ &= 2\sin a \sin b \\ &= 2\sin 4t \sin 40t. \end{aligned}$$

The graphic of the problem was obtained from `MAPLE`.

```
a:=4:b:=10*a:
opts:=scaling=constrained,axes=boxed,axesfont=[Courier,bold,16],
labelfont=[Courier,bold,16],thickness=4,color=[red,gray,black]:
 plot([sin(a*t)*sin(b*t),-sin(a*t),sin(a*t)],t=0..2*(2*Pi/a),opts);
```