Chapter 7: Laplace Theory

Background. Switches and Impulses

Laplace's method solves differential equations. It is the preferred method for solving equations containing switches or impulses.

Unit Step Define $u(t-a) = \begin{cases} 1 & t \ge a, \\ 0 & t < a. \end{cases}$. It is a **switch**, turned on at $t = a$.

Ramp Define $\mathbf{ramp}(t-a) = (t-a)u(t-a) = \begin{cases} t-a & t \ge a, \\ 0 & t < a. \end{cases}$, whose graph shape is a continuous **ramp** at 45-degree incline starting at $t = a$.

Unit Pulse Define $\mathbf{pulse}(t, a, b) = \begin{cases} 1 & a \le t < b, \\ 0 & \text{otherwise} \end{cases} = u(t-a) - u(t-b)$. The switch is **ON** at time $t = a$ and then **OFF** at time $t = b$.

Impulse of a Force

Define the **impulse** of an applied force $F(t)$ on time interval $a \le t \le b$ by the equation

$$\text{Impulse of } F = \int_a^b F(t)dt = \left(\frac{\int_a^b F(t)dt}{b-a}\right)(b-a) = \text{Average Force} \times \text{Duration Time}.$$

Dirac Unit Impulse

A Dirac impulse acts like a hammer hit, a brief injection of energy into a system. It is a special idealization of a real hammer hit, in which only the **impulse** of the force is deemed important, and not its magnitude nor duration.

Define the **Dirac Unit Impulse** by the equation $\delta(t-a) = \frac{du}{dt}(t-a)$, where $u(t-a)$ is the unit step. Symbol δ makes sense only under an integral sign, and the integral in question must be a generalized Riemann integral (definition pending), with new evaluation rules. Symbol δ is an abbreviation like **etc** or **e.g.**, because it abbreviates a paragraph of descriptive text.

- Symbol $M\delta(t-a)$ represents an ideal impulse of magnitude M at time $t = a$. Value M is the change in momentum, but $M\delta(t-a)$ contains no detail about the applied force or the duraction. A common force approximation for a hammer hit of very small duration $2h$ and impulse M is Dirac's approximation
$$F_h(t) = \frac{M}{2h}\mathbf{pulse}(t, a-h, a+h).$$
- Symbol δ is not manipulated as an ordinary function. It is a special modeling tool with rules for application and rules for algebraic manipulation.

THEOREM (Second Shifting Theorem). Let $f(t)$ and $g(t)$ be piecewise continuous and of exponential order. Then for $a \ge 0$,

$$\begin{aligned} e^{-as}\mathcal{L}(f(t)) &= \mathcal{L}\left(f(t)u(t)|_{t:=t-a}\right), \\ \mathcal{L}(g(t)u(t-a)) &= e^{-as}\mathcal{L}\left(g(t)|_{t:=t+a}\right). \end{aligned}$$

Sample Problem 2. Solve the following by Laplace methods.

(a) Forward table. Compute the Laplace integral for the unit step, ramp and pulse, in these special cases:

$$(1)\ \mathcal{L}(10u(t-\pi)) \quad (2)\ \mathcal{L}(\textbf{ramp}(t-2\pi)), \quad (3)\ \mathcal{L}(10\,\textbf{pulse}(t,3,5)).$$

(b) Backward table. Find $f(t)$ in the following special cases.

$$(1)\ \mathcal{L}(f)=\frac{5e^{-3s}}{s} \quad (2)\ \mathcal{L}(f)=\frac{e^{-4s}}{s^2} \quad (3)\ \mathcal{L}(f)=\frac{5}{s}\left(e^{-2s}-e^{-3s}\right).$$

Sample Problem 3. Solve the following Dirac Impulse and Second Shifting theorem problems.

(c) Forward table problems.

$$(1)\ \mathcal{L}(10\delta(t-\pi)), \quad (2)\ \mathcal{L}(5\delta(t-1)+10\delta(t-2)+15\delta(t-3)), \quad (2)\ \mathcal{L}((t-\pi)\delta(t-\pi)).$$

The sum of Dirac impulses in (2) is called an **impulse train**.

Solutions to sample problems 2 and 3

Solution (a). The forward second shifting theorem applies.

(1) $\mathcal{L}(10u(t-\pi)) = \mathcal{L}(g(t)u(t-a))$ where $g(t) = 10$ and $a = \pi$. Then $\mathcal{L}(10u(t-\pi)) = \mathcal{L}(g(t)u(t-a)) = e^{-as}\mathcal{L}\left(g(t)|_{t=t+a}\right) = e^{-\pi s}\mathcal{L}\left(10|_{t=t+\pi}\right) = \frac{10}{s}e^{-\pi s}$.

(2) $\mathcal{L}(\textsf{ramp}(t-2\pi)) = \mathcal{L}((t-2\pi)u(t-2\pi)) = \mathcal{L}\left(tu(t)|_{t=t-2\pi})\right) = e^{-2\pi s}\mathcal{L}(t) = \frac{1}{s^2}e^{-2\pi s}$.

(3) $\mathcal{L}(10\,\mathbf{pulse}(t,3,5)) = 10\,\mathcal{L}(u(t-3)-u(t-5)) = \frac{10}{s}(e^{3s}-e^{-5s})$.

Solution (b). Presence of an exponential e^{-as} signals step $u(t-a)$ in the answer, the main tool bing the backward second shifting theorem.

(1) $\mathcal{L}(f) = \frac{5e^{-3s}}{s} = e^{-3s}\frac{5}{s} = e^{-3s}\mathcal{L}(5) = \mathcal{L}(5u(t)|_{t=t+3}) = \mathcal{L}(5u(t-3))$. Lerch implies $f = 5u(t-3)$.

(2) $\mathcal{L}(f) = \frac{e^{-4s}}{s^2} = \frac{e{-as}}{\mathcal{L}}(t)$ where $a = 4$. Then $\mathcal{L}(f) = \frac{e{-as}}{\mathcal{L}}(t) = \mathcal{L}(tu(t)|_{t=t-a}) = \mathcal{L}((t-4)u(t-4)) = \mathcal{L}(\textsf{ramp}(t-4))$. Lerch implies $f = \textsf{ramp}(t-4)$.

(3) $\mathcal{L}(f) = e^{-2s}\frac{5}{s} - e^{-3s}\frac{5}{s} = \mathcal{L}(5u(t-2)) - \mathcal{L}(5u(t-3)) = \mathcal{L}(5\,\mathbf{pulse}(t,2,3))$. Lerch implies $f = 5\,\mathbf{pulse}(t,2,3)$.

Solution (c). The main result for Dirac unit impulse δ is the equation

$$\int_{)}^{\infty} g(t)\delta(t-a)dt = g(a),$$

valid for $g(t)$ continuous on $0 \le t < \infty$. When $g(t) = e^{-st}$, then the equation implies the Laplace formula $\mathcal{L}(\delta(t-a)) = e^{-as}$.

(1) $\mathcal{L}(10\delta(t-\pi)) = 10e^{-\pi s}$, by the displayed equation with $g(t) = 10e^{-st}$, or by using linearity and the formula $\mathcal{L}(\delta(t-a)) = e^{-as}$.

(2) $\mathcal{L}(5\delta(t-1) + 10\delta(t-2) + 15\delta(t-3)) = 5\,\mathcal{L}(\delta(t-1)) + 10\,\mathcal{L}(\delta(t-2)) + 15\,\mathcal{L}(\delta(t-3)) = 5e^{-s} + 10e^{-2s} + 15e^{-3s}$.

(3) $\mathcal{L}((t-\pi)\delta(t-2\pi)) = \int_0^\infty (t-\pi)e^{st}\delta(t-2\pi)dt = (t-\pi)e^{-st}|_{t=2\pi} = \pi e^{-2\pi s}$, using $g(t) = (t-\pi)e^{-st}$ and $a = 2\pi$ in the equation.

Sample Problem 4. **Experiment to Find the Transfer Function** $h(t)$

Consider a second order problem

$$ax''(t) + bx'(t) + cx(t) = f(t)$$

which by Laplace theory has a particular solution solution defined as the convolution of the transfer function $h(t)$ with the input $f(t)$,

$$x_p(t) = \int_0^t f(w)h(t-w)dw.$$

Examined in this problem is another way to find $h(t)$, which is the system response to a Dirac unit impulse with zero data. Then $h(t)$ is the solution of

$$ah''(t) + bh'(t) + ch(t) = \delta(t), \quad h(0) = h'(0) = 0.$$

The Problem. Assume a, b, c are constants and define $g(t) = \int_0^t h(w)dw$.

(a) Show that $g(0) = g'(0) = 0$, which means g has zero data.

(b) Let $u(t)$ be the unit step. Argue that g is the solution of

$$ag''(t) + bg'(t) + cg(t) = u(t), \quad g(0) = g'(0) = 0.$$

The fundamental theorem of calculus says that $h(t) = g'(t)$. Therefore, to compute the transfer function $h(t)$, find the response $g(t)$ to the unit step with zero data, followed by computing the derivative $g'(t)$, which equals $h(t)$.

The experimental impact is important. Turning on a switch creates a unit step, generally easier than designing a hammer hit.

(c) Illustrate the method for finding the transfer function $h(t)$ in the special case

$$x''(t) + 2x'(t) + 5x(t) = f(t).$$

Solutions to sample problem 4

(a) $g(0) = \int_0^0 h(w)dw = 0$, $g'(0) = h(0) = 0$.

(b) Let $u(t)$ be the unit step. Initial data was decided in part (a). The Laplace applied to $ag''(t) + bg'(t) + cg(t) = u(t)$ gives $(as^2 + bs + c)\,\mathcal{L}(g) = \mathcal{L}(u(t))$. Then $\mathcal{L}(g) = \mathcal{L}(h(t))\,\mathcal{L}(u(t)) = \mathcal{L}(h(t))\frac{1}{s}\,\mathcal{L}\left(\int_0^t h(r)du\right)$ by the **integral theorem**. Lerch's theorem then says $g(t) = \int_0^t h(r)dr$.

(c) For equation $x''(t) + 2x'(t) + 5x(t) = f(t)$ we replace $x(t)$ by $g(t)$ and $f(t)$ by the unit step $u(t)$, then solve $g''(t) + 2g'(t) + 5g(t) = u(t)$, obtaining $\mathcal{L}(g) = \frac{1}{s}\frac{1}{s^2+2s+5} = \mathcal{L}(\frac{1}{5} - \frac{1}{10}e^{-t}(2\cos(2t) + \sin(2t)))$. Then $g(t) = \frac{1}{5} - \frac{1}{10}e^{-t}(2\cos(2t) + \sin(2t))$ and $h(t) = g'(t) = \frac{1}{2}e^{-t}\sin(2t)$.