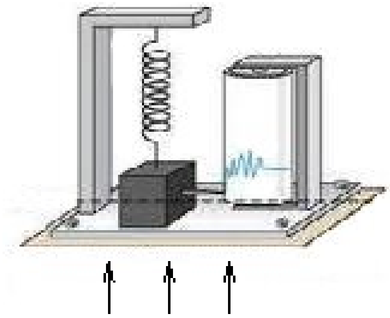
Sample Quiz 10

Sample Quiz 10, Problem 1. Vertical Motion Seismoscope

The 1875 **horizontal motion seismoscope** of F. Cecchi (1822-1887) reacted to an earthquake. It started a clock, and then it started motion of a recording surface, which ran at a speed of 1 cm per second for 20 seconds. The clock provided the observer with the earthquake hit time.



A Simplistic Vertical Motion Seismoscope

The apparently stationary heavy mass on a spring writes with the attached stylus onto a rotating drum, as the ground moves up. The generated trace is $x(t)$.

The motion of the heavy mass m in the figure can be modeled initially by a forced spring-mass system with damping. The initial model has the form

$$mx'' + cx' + kx = f(t)$$

where $f(t)$ is the vertical ground force due to the earthquake. In terms of the vertical ground motion $u(t)$, we write via Newton's second law the force equation $f(t) = -mu''(t)$ (compare to falling body $-mg$). The final model for the motion of the mass is then

$$(1) \qquad \begin{cases} x''(t) + 2\beta\Omega_0 x'(t) + \Omega_0^2 x(t) = -u''(t), \\ \dfrac{c}{m} = 2\beta\Omega_0, \quad \dfrac{k}{m} = \Omega_0^2, \\ x(t) = \text{center of mass position measured from equilibrium}, \\ u(t) = \text{vertical ground motion due to the earthquake.} \end{cases}$$

Terms **seismoscope, seismograph, seismometer** refer to the device in the figure. Some observations:

Slow ground movement means $x' \approx 0$ and $x'' \approx 0$, then (1) implies $\Omega_0^2 x(t) = -u''(t)$. The seismometer records ground acceleration.

Fast ground movement means $x \approx 0$ and $x' \approx 0$, then (1) implies $x''(t) = -u''(t)$. The seismometer records ground displacement.

A **release test** begins by starting a vibration with u identically zero. Two successive maxima $(t_1, x_1), (t_2, x_2)$ are recorded. This experiment determines constants β, Ω_0.

The objective of (1) is to determine $u(t)$, by knowing $x(t)$ from the seismograph.

The Problem.

(a) Explain how a **release test** can find values for β, Ω_0 in the model $x'' + 2\beta\Omega_0 x' + \Omega_0^2 x = 0$.

(b) Assume the seismograph trace can be modeled at time $t = 0$ (a time after the earthquake struck) by $x(t) = Ce^{-at}\sin(bt)$ for some positive constants C, a, b. Assume a release test determined $\beta = 6$ and $\Omega_0 = 10$. Explain how to find a formula for the ground motion $u(t)$, then provide a formula for $u(t)$, using technology.

Solution.

(a) A **release test** is an experiment which provides initial data $x(0) > 0$, $x'(0) = 0$ to the seismoscope mass. The model is $x'' + 2\beta\Omega_0 x' + \Omega_0^2 x = 0$ (ground motion zero). The recorder graphs $x(t)$ during the experiment, until two successive maxima $(t_1, x_1), (t_2, x_2)$ appear in the graph. This is enough information to find values for β, Ω_0.

In an under-damped oscillation, the characteristic equation is $(r+p)^2 + \omega^2 = 0$ corresponding to complex conjugate roots $-p \pm \omega i$. The phase-amplitude form is $x(t) = Ce^{-pt}\cos(\omega t - \alpha)$, with period $2\pi/\omega$.

The equation $x'' + 2\beta\Omega_0 x' + \Omega_0^2 x = 0$ has characteristic equation $(r+\beta)^2 + \Omega_0^2 = 0$. Therefore $x(t) = Ce^{-\beta t}\cos(\Omega_0 t - \alpha)$.

The period is $t_2 - t_1 = 2\pi/\Omega_0$. Therefore, Ω_0 is known. The maxima occur when the cosine factor is 1, therefore

$$\frac{x_2}{x_1} = \frac{Ce^{-\beta t_2}\cdot 1}{Ce^{-\beta t_1}\cdot 1} = e^{-\beta(t_2 - t_1)}.$$

This equation determines β.

(b) The equation $-u''(t) = x''(t) + 2\beta\Omega_0 x'(t) + \Omega_0^2 x(t)$ (the model written backwards) determines $u(t)$ in terms of $x(t)$. If $x(t)$ is known, then this is a quadrature equation for the ground motion $u(t)$.

For the example $x(t) = Ce^{-at}\sin(bt)$, $\beta = 6, \Omega_0 = 10$, the quadrature equation is

$$-u''(t) = x''(t) + 12x'(t) + 100x(t).$$

After substitution of $x(t)$, the equation becomes

$$-u''(t) = C\mathrm{e}^{-at}\left(\sin(bt)\,a^2 - \sin(bt)\,b^2 - 2\cos(bt)\,ab - 12\sin(bt)\,a + 12\cos(bt)\,b + 100\sin(bt)\right)$$

which can be integrated twice using Maple, for simplicity. All integration constants will be assumed zero. The answer:

$$u(t) = \frac{C\mathrm{e}^{-at}\left(12\,a^2 b + 12\,b^3 - 200\,ab\right)\cos(bt)}{\left(a^2+b^2\right)^2} - \frac{C\mathrm{e}^{-at}\left(a^4 + 2\,a^2 b^2 + b^4 - 12\,a^3 - 12\,ab^2 + 100\,a^2 - 100\,b^2\right)\sin(bt)}{\left(a^2+b^2\right)^2}$$

The Maple session has this brief input:

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de:=-diff(u(t),t,t) = diff(x(t),t,t) + 12*diff(x(t),t) + 100* x(t);
x:=t->C*exp(-a*t)*sin(b*t);
dsolve(de,u(t));subs(_C1=0,_C2=0,%);
```