

Chapters 1-2, Edwards–Penney

Sample Quiz1 Problem 1. An answer check for the differential equation and initial condition

$$\frac{dy}{dx} = -y(x) + 23, \quad y(0) = 5 \quad (1)$$

requires substitution of the candidate solution $y(x) = 23 - 18e^{-x}$ into the left side (LHS) and right side (RHS), then compare the expressions for equality for all symbols. The process of testing LHS = RHS applies to both the differential equation and the initial condition, making the answer check have **two** presentation panels. Complete the following:

1. Show the two panels in an answer check for initial value problem (1).
2. Relate (1) to a Newton cooling model for warming a 5 C apple to room temperature 23 C.

References. Edwards-Penney sections 1.1, 1.4, 1.5. Newton cooling in Serway and Vuille, *College Physics 9/E*, Brooks-Cole (2011), ISBN-10: 0840062060. Newton cooling differential equation $\frac{du}{dt} = -h(u(t) - u_1)$, Math 2280 slide Three Examples. Math 2280 slide on Answer checks.

Sample Quiz1 Problem 2. A 2-ft high institutional coffee maker serves coffee from an orifice 5 inches above the base of the cylindrical tank. The tank drains according to the Torricelli model

$$\frac{dy}{dx} = -0.02\sqrt{|y(x)|}, \quad y(0) = y_0. \quad (2)$$

Symbol $y(x) \geq 0$ is the tank coffee height in feet above the orifice at time x seconds, while $y_0 \geq 0$ is the coffee height at time $x = 0$.

Establish these facts about the physical problem.

1. If $y_0 = 0$, then $y(x)$ is not determined by the model. A physical explanation is expected, based on possible past tank levels. Numerical solutions are therefore technological nonsense.
2. If $y_0 > 0$, then the solution $y(x)$ is uniquely determined and computable by numerical software. Justify using Picard's existence-uniqueness theorem.
3. Solve equation (2) using separation of variables when y_0 is 19 inches, then numerically find the drain time (about 125 seconds).

References. Edwards-Penney, Picard's theorem 1 section 1.3 and Torricelli's Law section 1.4. Tank draining *Mathematica* demo at Wolfram Research. Carl Schaschke, *Fluid Mechanics: Worked Examples for Engineers*, The Institution of Chemical Engineers (2005), ISBN-10: 0852954980, Chapter 6. Math 2280 slide on Picard and Peano Theorems.

Sample Quiz 1

Problem 1, part 1.

panel 1.

$$\text{LHS} = \frac{dy}{dx}$$

$$= \frac{d}{dx}(23 - 18e^{-x})$$

$$= 0 + 18e^{-x}$$

$$\text{RHS} = -y + 23$$

$$= -(23 - 18e^{-x}) + 23$$

$$= 18e^{-x}$$

$\therefore \text{LHS} = \text{RHS}, \text{ DE } \checkmark$

panel 2.

$$\text{LHS} = y(0)$$

$$= (23 - 18e^{-x})|_{x=0}$$

$$= 23 - 18e^0$$

$$= 5$$

$$= \text{RHS}, \text{ IC } \checkmark$$

Problem 1, part 2.

Newton cooling is $u' = -h(u - u_1)$, $u(0) = u_0$. Changing $y \mapsto u$ and $x \mapsto t$ for the given DE + IC produces

$$\begin{cases} u' = -(u - 23), \\ u(0) = 5. \end{cases}$$

Then $h = +1$ is the cooling constant, $23 = u_1 =$ ambient temperature, $5 = u_0 =$ initial temperature. Then

$$\begin{cases} y(x) = u(t) = \text{apple temperature,} \\ 23 = u_1 = \text{wall thermometer temp,} \\ 5 = u_0 = \text{apple initial temp,} \\ -1 = h = \text{Newton cooling constant,} \\ x = t = \text{time.} \end{cases}$$

Sample Quiz 1

Problem 2, part 1.

The tank could drain any time $t_0 < 0$ in the past, meaning there is a solution $y(x)$ such that $y(x) > 0$ for $x < t_0$ and $y(x) = 0$ for $x \geq t_0$. In short, ∞ -many solutions. The model fails to determine a unique solution.

Problem 2, part 2.

If $y_0 > 0$, then $f(x, y) = -0.02\sqrt{|y|}$ and $\frac{\partial f}{\partial y} = -0.01|y|^{-1/2}$ on box $B = \{(x, y) : |x| \leq 10, \frac{1}{2}y_0 \leq y \leq 10\}$. Picard's Theorem says there is a smaller box $B_1 = \{(x, y) : |x| \leq H, \frac{1}{2}y_0 \leq y \leq 10\}$ on which a unique edge-to-edge solution $y(x)$ exists, $y(0) = y_0$.

Problem 2, part 3.

The IC is $y(0) = 19/12$ feet. Because $y > 0$, then $f(x, y) = F(x)G(y)$ with $F = -0.02$ and $G = \sqrt{y}$. Separation gives:

$$\frac{y'}{y^{1/2}} = -0.02$$

$$\int \frac{du}{u^{1/2}} = -0.02 \int dx, \quad u = y(x)$$

$$\frac{u^{1/2}}{1/2} = -0.02x + C_1$$

$$y^{1/2} = -0.01x + C$$

$$y = (-0.01x + C)^2$$

$$\sqrt{\frac{19}{12}} = (0 + C)$$

$$y = (-0.01x + \sqrt{19/12})^2$$

Drain time is x when $y = 0$, or $x = \frac{\sqrt{19/12}}{0.01} = 125.83$

Answer checked in Wolfram Alpha and Waterloo Maple.

method of quadrature
↓

square both sides,

$C = \sqrt{19/12}$ from 2 lines up ↑

Sample Quiz2 Problem 1.

Suppose a cup of hot chocolate has an initial temperature of $185^\circ F$ when freshly poured and the desired drinking temperature is $160^\circ F$. After 50 seconds in a room at $68^\circ F$, the temperature has cooled to $181^\circ F$. **Newton's Law of Cooling** applies to model the temperature $u(t)$ of the chocolate by the initial value problem

$$\frac{du}{dt} = -h(u(t) - 68), \quad u(0) = 185,$$

where $h > 0$ is the cooling constant, to be determined from supplied information.

1. Find an equation for the temperature $u(t)$ at any time t .
2. Find the Newton cooling constant h .
3. Determine the time required for the chocolate to cool to $160^\circ F$.

References. Edwards-Penney section 1.5. Serway and Vuille, *College Physics 9/E*, Brooks-Cole (2011), ISBN-10: 0840062060. An answer check might use *The Coffee Cooling Problem*, a Wolfram Demonstration Project contributed by S.M. Binder, which can be found at <http://demonstrations.wolfram.com/TheCoffeeCoolingProblem/>.

Credits. Created by Rebecca Terry, January 2014.

Sample Quiz2 Problem 2. Logistic growth $F(x) = rx(1 - x/K)$ can be used to describe the annual natural growth of a fish stock. Symbol $x(t)$ is the stock biomass in number of fish at the start of month t . The intrinsic growth rate is symbol r . The environmental carrying capacity in stock biomass terms is symbol K .

1. Assume a fish pond has carrying capacity $K = 780500$ and that 80% of the the fish survive to maturity. We'll assume 6 months to maturity and $r = 0.8$. Write in detail the no-harvesting model $x'(t) = F(x(t))$ and find the equilibrium values.
2. Assume constant harvesting H to give the model $x'(t) = F(x(t)) - H$. Use the quadratic formula from algebra to find the equilibrium points as a function of symbol $H \geq 0$. Then verify the following results.

If $H = 156100$, then there are two states: extinction for $x(0) < 390250$ and limiting population 390250 otherwise.

If $H > 156100$, then the extinction state is the only possibility.

If $H < 156100$, then there are two equilibria. The larger equilibrium population size is stable and the smaller is unstable. These numbers imply sustainable harvest for certain population sizes, but not all.

3. Assume a constant harvest rate H . Create two graphics of the population $x(t)$ over 36 months. The first uses a harvesting size H to show sustainable harvest. The second uses a different size H to show non-sustainable harvest. Handwritten plots are expected, or a computer plot, if you know how.

References. Edwards-Penney sections 2.1, 2.2. Course documents: Logistic Equation, Stability, Fish Farming and a logistic investigation in Malaysia by M.F. Laham 2012.

Sample Quiz 2 solutions plus maple code

Problem 1

Answers: (1) $u = 68 + 117e^{-ht}$

(2) $h = (-1/50) \ln\left(\frac{113}{117}\right) \approx 0.0006957$

(3) $t \approx 345.52$ seconds, about 6 min.

Details(1). Because $u = \text{degrees F}$ and $t = \text{seconds}$,
 then the model is $\begin{cases} u' = -h(u - 68), \\ u(0) = 185, u(50) = 181. \end{cases}$

The DE is solved by superposition $u = u_h + u_p$. The
 equilibrium solution is $u_p = 68$. Then $u_h = \frac{c}{\text{integ. factor}} =$
 $c e^{-ht}$, using standard linear form $u' + hu = 68h$.
 condition $u(0) = 185$ is used on $u = u_h + u_p = c e^{-ht} + 68$
 to evaluate $185 = c e^0 + 68$, then $c = 185 - 68 = 117$.

Details(2). Start with answer (1) and use $u(50) = 181$.

Then

$$181 = 68 + 117e^{-ht} \quad \text{when } t=50$$

$$113 = 117e^{-ht}$$

$$e^{-ht} = \frac{113}{117} \Rightarrow -ht = \ln\left(\frac{113}{117}\right)$$

$$\Rightarrow h = \frac{-1}{50} \ln\left(\frac{113}{117}\right)$$

Details(3). As in details (2),

$$160 = 68 + 117e^{-ht}$$

$$e^{-ht} = \frac{160 - 68}{117}$$

$$-ht = \ln\left(\frac{92}{117}\right) \quad \text{take log across eqn.}$$

$$t = \left(\frac{-1}{h}\right) \ln\left(\frac{92}{117}\right)$$

$$t = 50 \frac{\ln(92/117)}{\ln(113/117)} \approx 345.52 \text{ seconds}$$

Sample Quiz 2 solutions plus maple code

problem 2

Answers (1) $x' = 0.8x(1 - \frac{x}{780500})$, $x=0, 780500$

(2) $x^2 - Kx + \frac{HK}{r} = 0$, roots = $\frac{K}{2} \pm \frac{1}{2}\sqrt{K^2 - \frac{4HK}{r}}$

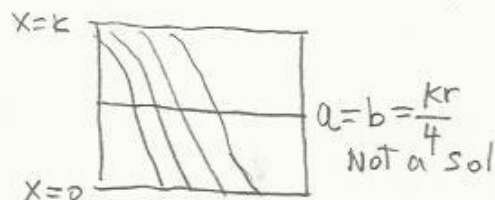


$$a = \frac{K}{2} + \frac{1}{2}\sqrt{D}$$

$$b = \frac{K}{2} - \frac{1}{2}\sqrt{D}$$

$$D = K^2 - \frac{4HK}{r}$$

Choose $H = \frac{Kr}{4} - 5000$



Choose $H = \frac{Kr}{4} + 5000$
All sols decrease to zero (and beyond), meaning extinction

Details (1). $x' = r x (1 - \frac{x}{K})$, substitute $r=0.8, k=780500$

Details (2). $x' = r x (1 - \frac{x}{K}) - H = r x - \frac{r}{K} x^2 - H$

$$x' = -\frac{r}{K} (x^2 - Kx + \frac{HK}{r})$$

Apply quadratic formula to $x^2 - Kx + \frac{HK}{r} = 0$ to find roots, reported in both (2), (3) above.

Details (3). A double real root is when the discriminant $D = K^2 - \frac{4HK}{r} = 0$, requiring $H = \frac{Kr}{4}$. For $H < \frac{Kr}{4}$, there are 2 real roots a, b as given in the answer. For $H > \frac{Kr}{4}$ there are no real roots, therefore $x' < 0$ and x decreases to zero (extinction) and beyond.

The larger root $a = \frac{K}{2} + \frac{1}{2}\sqrt{D}$ is a stable funnel.

The smaller root $b = \frac{K}{2} - \frac{1}{2}\sqrt{D}$ is an unstable spout.

When $D=0$, then $a=b = K/2$ is a node, when a, b are real, extinction for $x < b$, carrying capacity = a , sustainable harvest for $x > b$.

```
> F:=x->r*x*(1-x/K):G:=x->r*x*(1-x/K)-H:
> solve(G(x)=0,x);
```

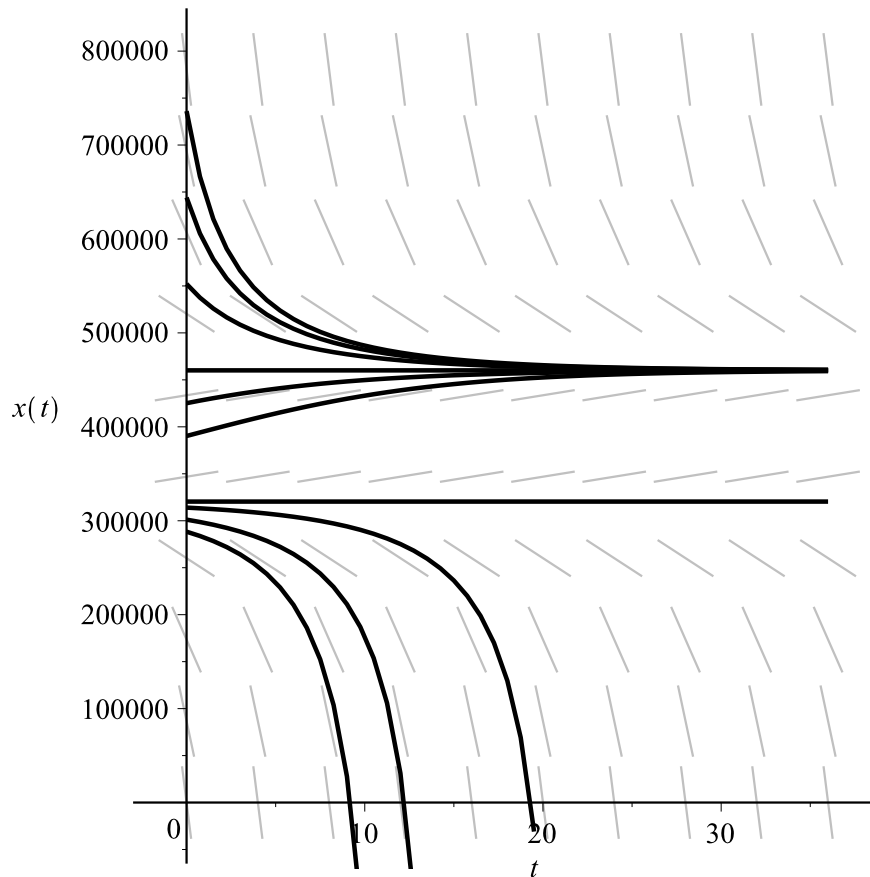
$$\frac{1}{2} \frac{Kr + \sqrt{K^2 r^2 - 4HKr}}{r}, \frac{1}{2} \frac{Kr - \sqrt{K^2 r^2 - 4HKr}}{r}$$

(1)

```
> de:=diff(x(t),t)=G(x(t)):r:=0.8:K:=780500:H0:=K*r/4:H:=H0-5000:
a:=K/2+(1/2)*sqrt(K^2-4*H*K/r);b:=K/2-(1/2)*sqrt(K^2-4*H*K/r);
ic:=[[0,0.9*b],[0,0.94*b],[0,0.98*b],[0,b],[0,(a+b)/2],[0,(3*a+b)
/4],
[0,a],[0,1.2*a],[0,1.4*a],[0,1.6*a]]:
opts:=dirfield=[10,10],arrows=line,color=gray,linecolor=black,
thickness=2:
> DEtools[DEplot](de,x(t),t=0..36,x=0..K,ic,opts);
```

$$a := 4.600935752 \cdot 10^5$$

$$b := 3.204064248 \cdot 10^5$$



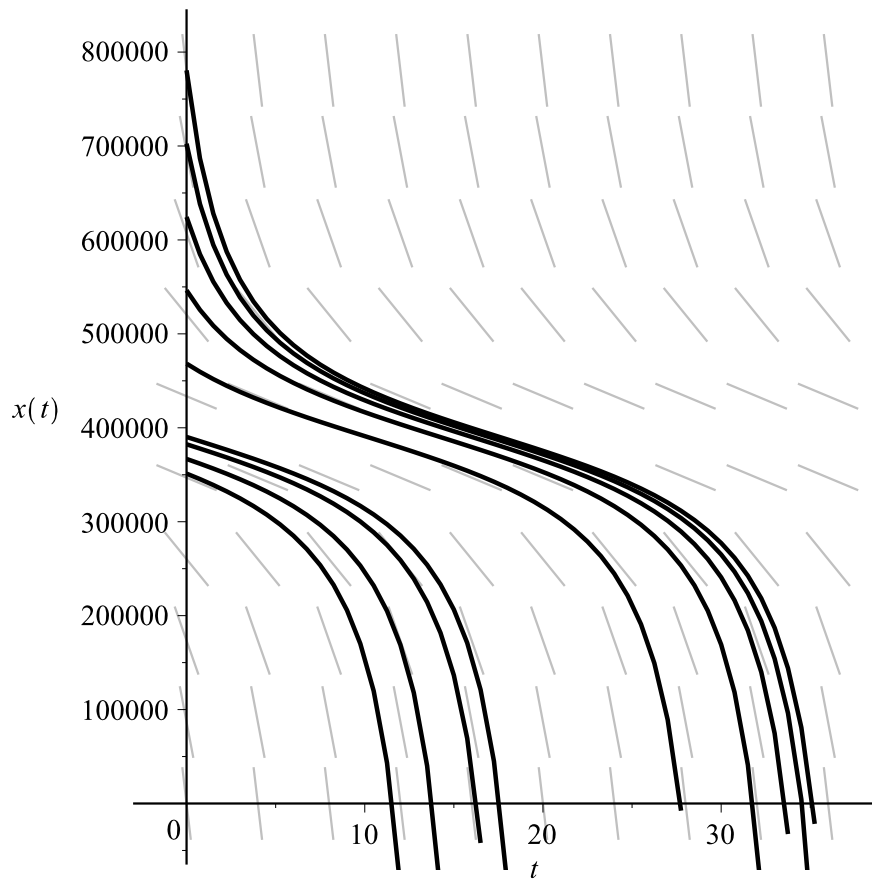
```
> de:=diff(x(t),t)=G(x(t)):
r:=0.8:K:=780500:H0:=K*r/4:
H:=H0+6000;a:=K/2;b:=K/2;
ic:=[[0,0.9*b],[0,0.94*b],[0,0.98*b],[0,b],[0,a],[0,1.2*a],[0,
1.4*a],
[0,1.6*a],[0,1.8*a],[0,2*a]]:
```

```
opts:=dirfield=[10,10],arrows=line,color=gray,linecolor=black,  
thickness=2:  
DEtools[DEplot](de,x(t),t=0..36,x=0..K,ic,opts);
```

$H := 1.621000000 \cdot 10^5$

$a := 390250$

$b := 390250$



All solutions decrease.

Sample Quiz3 Problem 1. A graphic called a **phase diagram** displays the behavior of all solutions of $u' = F(u)$. A **phase line diagram** is an abbreviation for a direction field on the vertical axis (u -axis). It consists of equilibrium points and signs of $F(u)$ between equilibria. A phase diagram can be created solely from a phase line diagram, using just three drawing rules:

1. Solutions don't cross.
2. Equilibrium solutions are horizontal lines $u = c$. All other solutions are increasing or decreasing.
3. A solution curve can be moved rigidly left or right to create another solution curve.

Use these tools on the equation $u' = u(u^2 - 4)$ to make a phase line diagram, and then make a phase diagram with at least 8 threaded solutions. Label the equilibria as stable, unstable, funnel, spout, node.

References. Edwards-Penney section 2.2. Course document on Stability,

Sample Quiz3 Problem 2. An autonomous differential equation $\frac{dy}{dx} = F(x)$ with initial condition $y(0) = y_0$ has a formal solution

$$y(x) = y_0 + \int_0^x F(u) du.$$

The integral may not be solvable by calculus methods. In this case, the integral is evaluated numerically to compute $y(x)$ or to plot a graphic. There are three basic numerical methods that apply, the rectangular rule (RECT), the trapezoidal rule (TRAP) and Simpson's rule (SIMP).

Apply the three methods for $F(x) = \sin(x^2)$ and $y_0 = 0$ using step size $h = 0.2$ from $x = 0$ to $x = 1$. Then fill in the blanks in the following table. Use technology if it saves time. Lastly, compare the four data sets in a plot, using technology.

x - values	0.0	0.2	0.4	0.6	0.8	1.0
y - to 10 digits	0.0	0.0026663619	0.02129435557	0.07133622797	0.1657380596	0.3102683017
y - RECT values	0.0	0.0	0.007997866838	<input type="text"/>	<input type="text"/>	0.2297554431
y - TRAP values	0.0	<input type="text"/>	0.02392968750	0.07508893150	<input type="text"/>	0.3139025416
y - SIMP values	0.0	0.002666288917	0.02129368017	<input type="text"/>	0.1657330636	<input type="text"/>

References. Edwards-Penney Sections 2.4, 2.5, 2.6, because methods Euler, Modified Euler and RK4 reduce to RECT, TRAP, SIMP methods when $f(x, y)$ is independent of y , i.e., an equation $y' = F(x)$. Course document on numerical solution of $y' = F(x)$ RECT, TRAP, SIMP methods. Wolfram Alpha at <http://www.wolframalpha.com/> can do the RECT rule and graphics with input string

`integrate sin(x^2) using left endpoint method with interval width 0.2 from x=0 to x=1`

Sample Quiz 3 Problem 1

$$u' = u(u^2 - 4)$$

phase line diagram, $F(u) = u(u^2 - 4) = u(u-2)(u+2)$

Equilibria are roots of $F(u) = 0$, then $u = 0, 2, -2$

$$F(-3) = \text{neg}, F(1) = \text{neg}$$

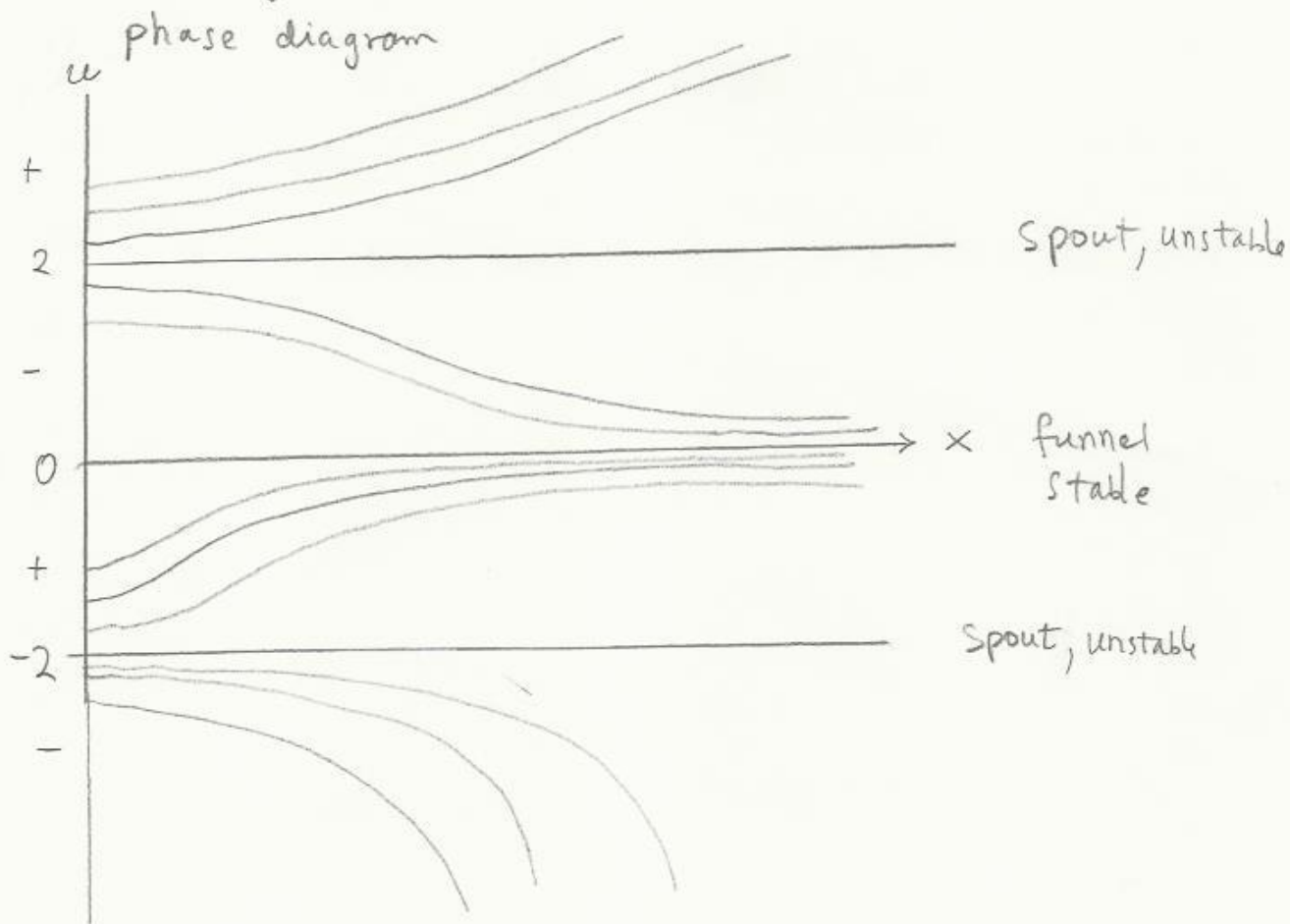
$$F(-1) = \text{pos}, F(3) = \text{pos}$$

Example: Sign - on $0 < u < 2$

because $F(1) = -3$ (F is one-signed for $0 < u < 2$)



The phase line diagram was constructed from F. we move it to the vertical u-axis, below. Sign + means increasing, - means decreasing.



Theorem

Between 2 adjacent equilibrium points, $F(u)$ is one-signed. Alternate: if $F(a) = F(b) = 0$ and F has no roots in $a < u < b$, then F is one-signed on $a < u < b$.

Sample Quiz 3 Problem 2 $\begin{cases} y' = \sin(x^2) \\ y(0) = 0 \end{cases}$

Solve numerically using Rect, Trap, Simp

RECT The approx formula is $y(x_0+h) = y(x_0) + h F(x_0)$, because $\int_a^b F dx \cong F(a)(b-a)$ for $b-a$ small. Then the supplied table implies

$$y(0.6) = y(0.4) + \int_{0.4}^{0.6} F dx \cong 0.007997866838 + (0.2) F(0.4)$$

$$= 0.007997866838 + (0.2) \sin(0.4^2)$$

$$= 0.03986150816$$

$$y(0.8) = 0.03986150816 + 0.2 \sin(0.6^2)$$

$$= 0.1103163548$$

TRAP The approx formula $\int_a^b F dx \cong (b-a) \frac{F(a)+F(b)}{2}$ implies that

$y(x_0+h) = y(x_0) + \frac{h}{2} (F(x_0) + F(x_0+h))$. The Table implies

$$y(0.2) = y(0.0) + \frac{0.2}{2} (F(0) + F(0.2))$$

$$= 0 + 0.1 (\sin(0) + \sin(0.2^2))$$

$$= 0.003998933419$$

$$y(0.8) = y(0.6) + 0.1 (F(0.6) + F(0.8))$$

$$= 0.07508893150 + 0.1 (\sin(0.6^2) + \sin(0.8^2))$$

$$= 0.1700358989$$

RK4 use $\int_a^b F dx \cong \frac{(b-a)}{6} (F(a) + 4F(\frac{a+b}{2}) + F(b))$ to get

$y(x_0+h) = y(x_0) + \frac{h}{6} (F(x_0) + 4F(x_0+h/2) + F(x_0+h))$. The Table implies

$$y(0.6) = y(0.4) + \frac{0.2}{6} (F(0.4) + 4F(0.5) + F(0.6))$$

$$= 0.2129368017 + \frac{0.1}{3} (\sin(0.4^2) + 4\sin(0.5^2) + \sin(0.6^2))$$

$$= 0.07133395608$$

$$y(1.0) = y(0.8) + \frac{0.1}{3} (\sin(0.8^2) + 4\sin(0.9^2) + \sin(1.0^2))$$

$$= 0.3102602343$$

Solution Key

0.0039989334

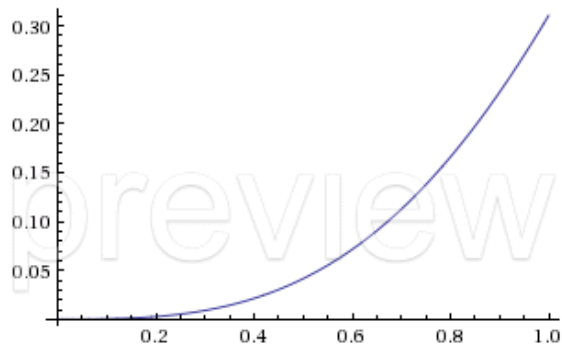
0.039861508

0.110316354

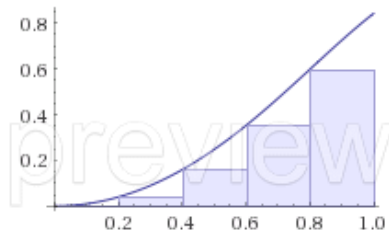
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0.07133395608

0.3102602343



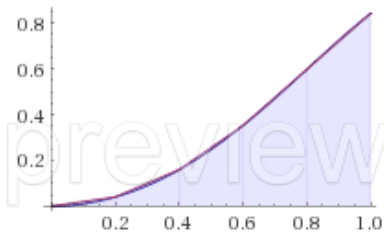
Computed by Wolfram|Alpha



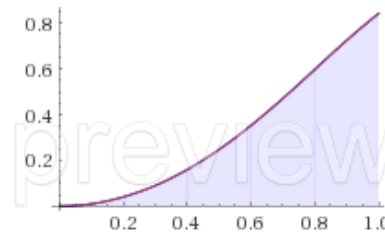
Computed by Wolfram|Alpha

10-digit integral of $\sin(x^2)$

RECT rule plot of $y'=\sin(x^2)$, $h=0.2$



Computed by Wolfram|Alpha



Computed by Wolfram|Alpha

TRAP rule plot of $y'=\sin(x^2)$, $h=0.2$

SIMP rule plot of $y'=\sin(x^2)$, $h=0.2$

Sample Quiz 4

Sample Quiz4 Problem 1. The velocity of a crossbow arrow fired upward from the ground is given at different times in the following table.

Time t in seconds	Velocity $v(t)$ in ft/sec	Location
0.000	50	Ground
1.413	0	Maximum
2.980	-45	Near Ground Impact



- (a) The velocity $v(t)$ can be approximated by a quadratic polynomial

$$z(t) = at^2 + bt + c$$

which reproduces the table data. Find three equations for the coefficients a, b, c . Then solve for them to obtain $a \approx 2.238$, $b \approx -38.55$, $c = 50$.

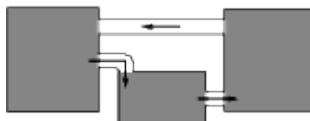
- (b) Assume a linear drag model $v' = -32 - \rho v$. Substitute the polynomial answer $v = z(t)$ of (a) into this differential equation, then substitute $t = 0$ and solve for $\rho \approx 0.131$.
- (c) Solve the model $w' = -32 - \rho w$, $w(0) = 50$ to get $w(t) = -\frac{32}{\rho} + \left(50 + \frac{32}{\rho}\right) e^{-\rho t}$. Substitute $\rho = 0.131$. Then $w(t) = -244.2748092 + 294.2748092 e^{-0.131 t}$ is an exponential model for linear drag which might reproduce the crossbow data.
- (d) Compare $w(t)$ and $z(t)$ in a plot. Comment on the plot and what it means. Bear in mind that $w(t)$ is an exponential model while $z(t)$ is a quadratic model. Neither of them are the true velocity $v(t)$ which produced the crossbow data.

References. Edwards-Penney sections 2.3, 3.1, 3.2. Course documents on Linear algebraic equations and Newton kinematics.

Sample Quiz4 Problem 2. Consider the system of differential equations

$$\begin{aligned} x_1' &= -\frac{1}{6}x_1 && + \frac{1}{6}x_3, \\ x_2' &= \frac{1}{6}x_1 && - \frac{1}{3}x_2, \\ x_3' &= && \frac{1}{3}x_2 - \frac{1}{6}x_3, \end{aligned}$$

for the amounts x_1, x_2, x_3 of salt in recirculating brine tanks, as in the figure:



Recirculating Brine Tanks A, B, C

The volumes are 60, 30, 60 for A, B, C, respectively.

The steady-state salt amounts in the three tanks are found by formally setting $x_1' = x_2' = x_3' = 0$ and then solving for the symbols x_1, x_2, x_3 . Solve the corresponding linear system of algebraic equations to obtain the answer $x_1 = x_3 = 2c$, $x_2 = c$, which means the total amount of salt is uniformly distributed in the tanks in ratio 2 : 1 : 2.

References. Edwards-Penney sections 3.1, 3.2, 7.3 Figure 5. Course documents on Linear algebraic equations and Systems and Brine Tanks.

Sample Quiz 4 Solutions

Problem 1

(a) Let $t_1 = 1.413$, $t_2 = 2.98$. Use $at^2 + bt + c = v(t)$ for $t = 0, t_1, t_2$ to obtain the system

$$\begin{cases} a \cdot 0^2 + b \cdot 0 + c = 50 \\ a \cdot t_1^2 + b \cdot t_1 + c = 0 \\ a \cdot t_2^2 + b \cdot t_2 + c = -45 \end{cases}$$

Then $\boxed{c=50}$. The 3×3 system reduces to a 2×2 system

$$\begin{cases} a t_1^2 + b t_1 = -50 \\ a t_2^2 + b t_2 = -95 \end{cases}$$

$$\begin{cases} a + b/t_1 = -50/t_1^2 & \text{mult}(1, 1/t_1^2) \\ a t_2^2 + b t_2 = -95 \end{cases}$$

$$\begin{cases} a + b/t_1 = -50/t_1^2 \\ 0 + b \cdot t_3 = -95 + \frac{50 t_2^2}{t_1^2} & \text{combo}(1, 2, -t_2^2) \\ & \text{where } t_3 = t_2 - \frac{t_2^2}{t_1} \end{cases}$$

$$\text{Then } \boxed{b} = \frac{1}{t_2 - t_1} \left(-\frac{50 t_2}{t_1} + 95 \frac{t_1}{t_2} \right) = \boxed{-38.54760463}$$

$$\boxed{a} = \frac{1}{t_2 - t_1} \left(\frac{50}{t_1} - \frac{95}{t_2} \right) = \boxed{2.23772148}$$

The example gives evidence for why technology is used on systems of equations. For 2-digit accuracy, it is less hand work, and quite fast with a calculator.

(b) Substitution gives $2at + b = -32 - p(at^2 + bt + c)$,
 Then $t=0$ implies $b = -32 - pc$. Calculator gives
 $p = (-32 - b)/c = 0.1309520926 \cong 0.131$

(c) $w = \text{equil sol} + \frac{c_1}{\text{integ factor}} = \frac{-32}{p} + \frac{c_1}{e^{pt}}$. Then $w(0) = 50$
 implies $c_1 = 50 + 32/p$.

(d) A good plot is $|v(t) - w(t)|$ on $0 \leq t \leq 3$. It shows max error of 0.3. typo: v(t) above should be z(t)

Sample Quiz 4 Solutions

Problem 2 System $\begin{cases} -\frac{1}{6}x_1 + \frac{1}{6}x_3 = 0 \\ \frac{1}{6}x_1 - \frac{1}{3}x_2 = 0 \\ \frac{1}{2}x_2 - \frac{1}{6}x_3 = 0 \end{cases}$ has augmented

matrix equal to

$$\left(\begin{array}{ccc|c} -\frac{1}{6} & 0 & \frac{1}{6} & 0 \\ \frac{1}{6} & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{6} & 0 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 2 & -1 & 0 \end{array} \right) \quad \text{mult}(1,6), \text{mult}(2,6), \text{mult}(3,6)$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{array} \right) \quad \text{combo}(1,2,1)$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad \text{Combo}(2,3,1)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad \text{mult}(1,-1), \text{mult}(2,-1)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad \text{mult}(2, \frac{1}{2}) \quad \text{Last frame}$$

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 - \frac{1}{2}x_3 = 0 \\ 0 = 0 \end{cases}$$

reduced echelon system
last frame algorithm applies

Answer

$$\begin{cases} x_1 = t_1 \\ x_2 = \frac{1}{2}t_1 \\ x_3 = t_1 \end{cases} \quad -\infty < t_1 < \infty$$

Let $t_1 = 2c$

Then

$$\begin{cases} x_1 = 2c \\ x_2 = c \\ x_3 = 2c \end{cases}$$

Sample Quiz 5

Sample Quiz 5, Problem 1. Harmonic Vibration

A mass of $m = 250$ grams attached to a spring of Hooke's constant k undergoes free undamped vibration. At equilibrium, the spring is stretched 25 cm by a force of 8 Newtons. At time $t = 0$, the spring is stretched 0.5 m and the mass is set in motion with initial velocity 5 m/s directed away from equilibrium. Find:

- (a) The numerical value of Hooke's constant k .
- (b) The initial value problem for vibration $x(t)$.

Solution

(a): Hooke's law Force= k (elongation) is applied with force 8 Newtons and elongation $25/100 = 1/4$ meter. Equation $8 = k(1/4)$ implies $k = 32$ N/m.

(b): Given $m = 250/1000$ kg and $k = 32$ N/m from part (a), then the free vibration model $mx'' + kx = 0$ becomes $\frac{1}{4}x'' + 32x = 0$. Initial conditions are $x(0) = 0.5$ m and $x'(0) = 5$ m/s. The initial value problem is

$$\begin{cases} \frac{d^2x}{dt^2} + 128x = 0, \\ x(0) = 0.5, \\ x'(0) = 5. \end{cases}$$

Sample Quiz 5, Problem 2. Phase-Amplitude Conversion

Write the vibration equation

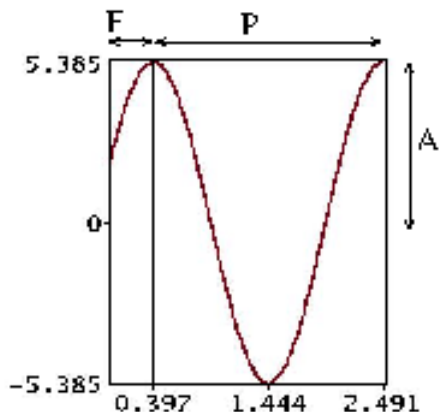
$$x(t) = 2 \cos(3t) + 5 \sin(3t)$$

in phase-amplitude form $x = A \cos(\omega t - \alpha)$. Create a graphic of $x(t)$ with labels for period, amplitude and phase shift.

Solution

The answer and the graphic appear below.

$$x(t) = \sqrt{29} \cos(3t - 1.190289950) = \sqrt{29} \cos(3(t - 0.3967633167)).$$



Harmonic Oscillation

The graph of $2 \cos(3t) + 5 \sin(3t)$. It has amplitude $A = \sqrt{29} = 5.385$, period $P = 2\pi/3$ and phase shift $F = 0.3967633167$. The graph is on $0 \leq t \leq P + F$.

Algebra Details. The plan is to re-write $x(t)$ in the form $x(t) = A \cos(\omega t - \alpha)$, called the phase-amplitude form of the harmonic oscillation.

Start with $x(t) = 2 \cos(3t) + 5 \sin(3t)$. Trig identity $x(t) = A \cos(\omega t - \alpha) = A \cos(\alpha) \cos(\omega t) + A \sin(\alpha) \sin(\omega t)$ causes the definitions

$$\omega = 3, \quad A \cos(\alpha) = 2, \quad A \sin(\alpha) = 5.$$

The Pythagorean identity $\cos^2 \alpha + \sin^2 \alpha = 1$ implies $A^2 = 2^2 + 5^2 = 29$ and then the amplitude is $A = \sqrt{29}$. Because $\cos \alpha = 2/A$, $\sin \alpha = 5/A$, then both the sine and cosine are positive, placing angle α in quadrant I. Divide equations $\cos \alpha = 2/A$, $\sin \alpha = 5/A$ to obtain $\tan(\alpha) = 5/2$, which by calculator implies $\alpha = \arctan(5/2) = 1.190289950$ radians or 68.19859051 degrees. Then $x(t) = A \cos(\omega t - \alpha) = \sqrt{29} \cos(3t - 1.190289950)$.

Computer Details. Either equation for $x(t)$ can be used to produce a computer graphic. A hand-drawn graphic would use only the phase-amplitude form. The period is $P = 2\pi/\omega = 2\pi/3$. The amplitude is $A = \sqrt{29} = 5.385164807$ and the phase shift is $F = \alpha/\omega = 0.3967633167$. The graph is on $0 \leq t \leq P + F$.

Maple

```
F:=evalf(arctan(5/2)/3); P:=2*Pi/3;A:=sqrt(29);
X:=t->2*cos(3*t)+5*sin(3*t);
opts:=xtickmarks=[0,F,P/2+F,P+F],ytickmarks=[-A,0,A],
      axes=boxed,thickness=3,labels=["",""];
plot(X(t),t=0..P+F,opts);
```

Sample Quiz 5, Problem 3. Undamped Spring-Mass System

A mass of 6 Kg is attached to a spring that elongates 20 centimeters due to a force of 12 Newtons. The motion starts at equilibrium with velocity -5 m/s. Find an equation for $x(t)$ using the free undamped vibration model $mx'' + kx = 0$.

Solution

The answer is $x(t) = -\sqrt{\frac{5}{2}} \sin(\sqrt{10}t)$.

The mass is $m = 6$ kg. Hooke's law $F = kx$ is applied with $F = 12$ N and $x = 20/100$ m. Then Hooke's constant is $k = 60$ N/m. Initial conditions are $x(0) = 0$ m (equilibrium) and $x'(0) = -5$ m/s.

The Model.

$$\begin{cases} 6 \frac{d^2x}{dt^2} + 60x &= 0, \\ x(0) &= 0, \\ x'(0) &= -5. \end{cases}$$

Solving the Model.

The characteristic equation $6r^2 + 60 = 0$ is solved for $r = \pm i\sqrt{10}$, then the Euler solution atoms are $\cos(\sqrt{10}t)$, $\sin(\sqrt{10}t)$ and we write the general solution as

$$x(t) = c_1 \cos(\sqrt{10}t) + c_2 \sin(\sqrt{10}t).$$

The task remaining is determination of constants c_1, c_2 subject to initial conditions $x(0) = 0$, $x'(0) = -5$. The linear algebra problem uses the derivative formula

$$x'(t) = -\sqrt{10}c_1 \sin(\sqrt{10}t) + \sqrt{10}c_2 \cos(\sqrt{10}t).$$

The 2×2 system of linear algebraic equations for c_1, c_2 is obtained from the two equations $x(0) = 0$, $x'(0) = -5$ as follows.

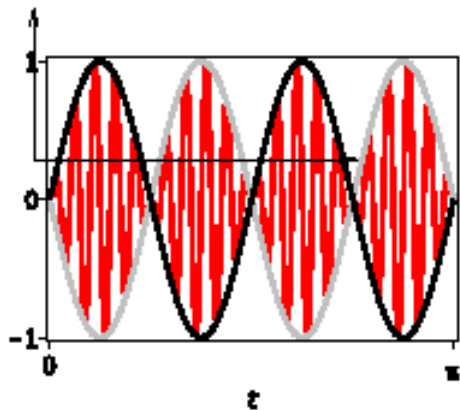
$$\begin{cases} \cos(0)c_1 + \sin(0)c_2 &= 0, & \text{Equation } x(0) = 0 \\ -\sqrt{10}\sin(0)c_1 + \sqrt{10}\cos(0)c_2 &= -5, & \text{Equation } x'(0) = -5 \end{cases}$$

Because $\cos(0) = 1$, $\sin(0) = 0$, then $c_1 = 0$ and $c_2 = -5/\sqrt{10} = -\sqrt{5}/2$. Insert answers c_1, c_2 into the general solution to find the answer to the initial value problem

$$x(t) = -\sqrt{\frac{5}{2}} \sin(\sqrt{10}t).$$

Sample Quiz 5, Problem 4. Beats

The physical phenomenon of **beats** refers to the periodic interference of two sound waves of slightly different frequencies. A destructive interference occurs during a very brief interval, so our impression is that the sound periodically stops, only briefly, and then starts again with a beat, a section of sound that is instantaneously loud again. Human heartbeat uses the same terminology. Our pulse rate is 40 – 100 **beats** per minute at rest. An illustration of the graphical meaning appears in the figure below.



Beats

Shown in red is a periodic oscillation $x(t) = 2 \sin 4t \sin 40t$ with rapidly-varying factor $\sin 40t$ and the two slowly-varying envelope curves $x_1(t) = 2 \sin 4t$ (black), $x_2(t) = -2 \sin 4t$ (grey).

The undamped, forced spring-mass problem $x'' + 1296x = 640 \cos(44t)$, $x(0) = x'(0) = 0$ has by trig identities the solution $x(t) = \cos(36t) - \cos(44t) = 2 \sin 4t \sin 40t$.

A key example is piano tuning. A tuning fork is struck, then the piano string is tuned until the beats are not heard. The number of beats per second (unit Hz) is approximately the frequency difference between the two sources, e.g., two tuning forks of frequencies 440 Hz and 437 Hz would produce 3 beats per second.

The average human ear can detect beats only if the two interfering sound waves have a frequency difference of about 7 Hz or less. Ear-tuned pianos are subject to the same human ear limitations. Two piano keys are more than 7 Hz apart, even for a badly tuned piano, which is why simultaneously struck piano keys are heard as just one sound (no beats).

The beat we hear corresponds to maxima in the figure. We see not the two individual sound waves, but their **superposition**. When the tuning fork and the piano string have the same exact frequency ω , then the figure would show a simple harmonic wave, because the two sounds would **superimpose** to a graph that looks like $\cos(\omega t - \alpha)$.

The origin of the phenomenon of **beats** can be seen from the formula

$$x(t) = 2 \sin at \sin bt.$$

There is no sound when $x(t) \approx 0$: this is when destructive interference occurs. When a is small compared to b , e.g., $a = 4$ and $b = 40$, then there are long intervals between the zeros of $A(t) = 2 \sin at$, at which destructive interference occurs. Otherwise, the amplitude of the sound wave is the average value of $A(t)$, which is 1. The sound stops at a zero of $A(t)$ and then it is rapidly loud again, causing the beat.

The Problem. Solve the initial value problem

$$x'' + 1296x = 640 \cos(44t), \quad x(0) = x'(0) = 0$$

by undetermined coefficients and linear algebra, obtaining the solution $x(t) = \cos(36t) - \cos(44t)$. Then show the trig details for $x(t) = 2 \sin(4t) \sin(40t)$. Finally, graph $x(t)$ and its slowly varying envelope curves on $0 \leq t \leq \pi$.

Solution to Problem 4.

The trial solution for undetermined coefficients has the form $x = d_1 \cos(44t) + d_2 \sin(44t)$. This is the Rule I trial solution. But Rule II does not modify the trial solution, because the Euler solution atoms $\cos(44t), \sin(44t)$ are not solutions of the homogeneous equation $x'' + 1296x = 0$.

Insert the trial solution into $x'' + 1296x = 640 \cos(44t)$ to obtain the equation

$$(1296 - 44^2)d_1 \cos(44t) + (1296 - 44^2)d_2 \sin(44t) = 640 \cos(44t).$$

Then matching atoms across the equal sign implies $d_1 = 640/(1296 - 44^2) = -1$, $d_2 = 0$. The particular solution is the trial solution with $d_1 = -1$, $d_2 = 0$. The formula obtained so far is

$$x_p(t) = -\cos(44t).$$

The homogeneous solution $x_h(t)$ is found from the characteristic equation $r^2 + 1296 = 0$, with complex conjugate roots $r = \pm 36i$. Then

$$x_h(t) = c_1 \cos(36t) + c_2 \sin(36t).$$

The initial conditions $x(0) = x'(0) = 0$ are used together with the general solution and its derivative

$$\begin{aligned} x(t) &= x_h(t) + x_p(t) = c_1 \cos(36t) + c_2 \sin(36t) - \cos(44t) \\ x'(t) &= x'_h(t) + x'_p(t) = -36c_1 \sin(36t) + 36c_2 \cos(36t) + 44 \sin(44t) \end{aligned}$$

to obtain the 2×2 linear algebraic system of equations

$$\begin{cases} \cos(0)c_1 + \sin(0)c_2 = \cos(0), & \text{Equation } x(0) = 0 \\ -36 \sin(0)c_1 + 36 \cos(0)c_2 = -44 \sin(0). & \text{Equation } x'(0) = 0 \end{cases}$$

Then $c_1 = 1$, $c_2 = 0$ and $x(t) = \cos(36t) - \cos(44t)$.

The trig identities $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$, $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ are subtracted to give $\cos(a-b) - \cos(a+b) = 2\sin(a)\sin(b)$. Let $a-b = 36t$, $a+b = 44t$ and solve for $a = 40t$, $b = 4t$. Then

$$\begin{aligned} x(t) &= \cos(36t) - \cos(44t) \\ &= \cos(a-b) - \cos(a+b) \\ &= 2 \sin a \sin b \\ &= 2 \sin 4t \sin 40t. \end{aligned}$$

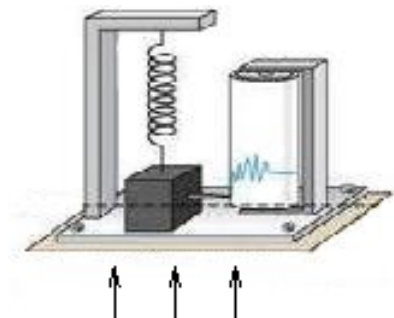
The graphic of the problem was obtained from MAPLE.

```
a:=4:b:=10*a:
opts:=scaling=constrained,axes=boxed,axesfont=[Courier,bold,16],
labelfont=[Courier,bold,16],thickness=4,color=[red,gray,black]:
plot([sin(a*t)*sin(b*t),-sin(a*t),sin(a*t)],t=0..2*(2*Pi/a),opts);
```

Sample Quiz 6

Sample Quiz 6, Problem 1. Vertical Motion Seismoscope

The 1875 **horizontal motion seismoscope** of F. Cecchi (1822-1887) reacted to an earthquake. It started a clock, and then it started motion of a recording surface, which ran at a speed of 1 cm per second for 20 seconds. The clock provided the observer with the earthquake hit time.



A Simplistic Vertical Motion Seismoscope

The apparently stationary heavy mass on a spring writes with the attached stylus onto a rotating drum, as the ground moves up. The generated trace is $x(t)$.

The motion of the heavy mass m in the figure can be modeled initially by a forced spring-mass system with damping. The initial model has the form

$$mx'' + cx' + kx = f(t)$$

where $f(t)$ is the vertical ground force due to the earthquake. In terms of the vertical ground motion $u(t)$, we write via Newton's second law the force equation $f(t) = -mu''(t)$ (compare to falling body $-mg$). The final model for the motion of the mass is then

$$\begin{cases} x''(t) + 2\beta\Omega_0 x'(t) + \Omega_0^2 x(t) = -u''(t), \\ \frac{c}{m} = 2\beta\Omega_0, \quad \frac{k}{m} = \Omega_0^2, \\ x(t) = \text{center of mass position measured from equilibrium,} \\ u(t) = \text{vertical ground motion due to the earthquake.} \end{cases} \quad (1)$$

Terms **seismoscope**, **seismograph**, **seismometer** refer to the device in the figure. Some observations:

Slow ground movement means $x' \approx 0$ and $x'' \approx 0$, then (1) implies $\Omega_0^2 x(t) = -u''(t)$. The seismometer records ground acceleration.

Fast ground movement means $x \approx 0$ and $x' \approx 0$, then (1) implies $x''(t) = -u''(t)$. The seismometer records ground displacement.

A **release test** begins by starting a vibration with u identically zero. Two successive maxima $(t_1, x_1), (t_2, x_2)$ are recorded. This experiment determines constants β, Ω_0 .

The objective of (1) is to determine $u(t)$, by knowing $x(t)$ from the seismograph.

The Problem.

(a) Explain how a **release test** can find values for β, Ω_0 in the model $x'' + 2\beta\Omega_0 x' + \Omega_0^2 x = 0$.

(b) Assume the seismograph trace can be modeled at time $t = 0$ (a time after the earthquake struck) by $x(t) = Ce^{-at} \sin(bt)$ for some positive constants C, a, b . Assume a release test determined $2\beta\Omega_0 = 12$ and $\Omega_0^2 = 100$. Explain how to find a formula for the ground motion $u(t)$, then provide a formula for $u(t)$, using technology.

Solution.

(a) A **release test** is an experiment which provides initial data $x(0) > 0$, $x'(0) = 0$ to the seismoscope mass. The model is $x'' + 2\beta\Omega_0x' + \Omega_0^2x = 0$ (ground motion zero). The recorder graphs $x(t)$ during the experiment, until two successive maxima $(t_1, x_1), (t_2, x_2)$ appear in the graph. This is enough information to find values for β, Ω_0 .

In an under-damped oscillation, the characteristic equation is $(r + p)^2 + \omega^2 = 0$ corresponding to complex conjugate roots $-p \pm \omega i$. The phase-amplitude form is $x(t) = Ce^{-pt} \cos(\omega t - \alpha)$, with period $2\pi/\omega$.

The equation $x'' + 2\beta\Omega_0x' + \Omega_0^2x = 0$ has characteristic equation $(r + \beta)^2 + \Omega_0^2 = 0$. Therefore $x(t) = Ce^{-\beta t} \cos(\Omega_0 t - \alpha)$.

The period is $t_2 - t_1 = 2\pi/\Omega_0$. Therefore, Ω_0 is known. The maxima occur when the cosine factor is 1, therefore

$$\frac{x_2}{x_1} = \frac{Ce^{-\beta t_2} \cdot 1}{Ce^{-\beta t_1} \cdot 1} = e^{-\beta(t_2 - t_1)}.$$

This equation determines β .

(b) The equation $-u''(t) = x''(t) + 2\beta\Omega_0x'(t) + \Omega_0^2x(t)$ (the model written backwards) determines $u(t)$ in terms of $x(t)$. If $x(t)$ is known, then this is a quadrature equation for the ground motion $u(t)$.

For the example $x(t) = Ce^{-at} \sin(bt)$, $2\beta\Omega_0 = 12$, $\Omega_0^2 = 100$, then the quadrature equation is

$$-u''(t) = x''(t) + 12x'(t) + 100x(t).$$

After substitution of $x(t)$, the equation becomes

$$-u''(t) = Ce^{-at} \left(\sin(bt) a^2 - \sin(bt) b^2 - 2 \cos(bt) ab - 12 \sin(bt) a + 12 \cos(bt) b + 100 \sin(bt) \right)$$

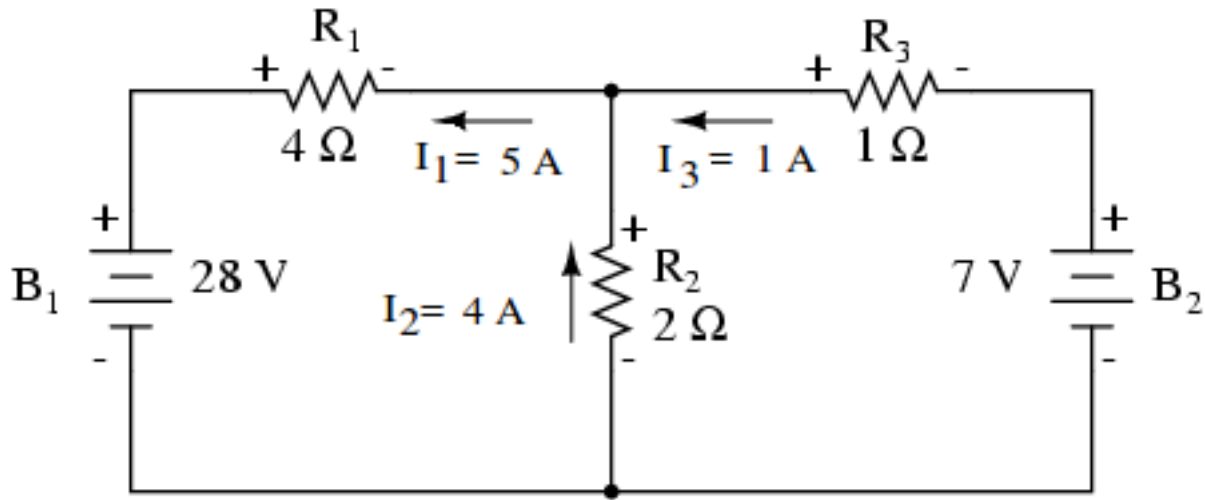
which can be integrated twice using Maple, for simplicity. All integration constants will be assumed zero. The answer:

$$u(t) = \frac{Ce^{-at} (12 a^2 b + 12 b^3 - 200 ab) \cos(bt)}{(a^2 + b^2)^2} - \frac{Ce^{-at} (a^4 + 2 a^2 b^2 + b^4 - 12 a^3 - 12 ab^2 + 100 a^2 - 100 b^2) \sin(bt)}{(a^2 + b^2)^2}$$

The Maple session has this brief input:

```
de:=-diff(u(t),t,t) = diff(x(t),t,t) + 12*diff(x(t),t) + 100* x(t);
x:=t->C*exp(-a*t)*sin(b*t);
dsolve(de,u(t));subs(_C1=0,_C2=0,%);
```

Sample Quiz6 Problem 2. Resistive Network with 2 Loops and DC Sources.



The **Branch Current Method** can be used to find a 3×3 linear system for the **branch currents** I_1, I_2, I_3 .

$$\begin{array}{rcll} I_1 - I_2 - I_3 & = & 0 & \text{KCL, upper node} \\ 4I_1 + 2I_2 & = & 28 & \text{KVL, left loop} \\ 2I_2 - I_3 & = & 7 & \text{KVL, right loop} \end{array}$$

Symbol **KCL** means *Kirchhoff's Current Law*, which says the algebraic sum of the currents at a node is zero. Symbol **KVL** means *Kirchhoff's Voltage Law*, which says the algebraic sum of the voltage drops around a closed loop is zero.

(a) Solve the equations to verify the currents reported in the figure: $I_1 = 5, I_2 = 4, I_3 = 1$ Amperes.

(b) Compute the voltage drops across resistors R_1, R_2, R_3 . Answer: 20, 8, 1 volts.

References. Edwards-Penney 3.7, electric circuits. All About Circuits Volume I – DC, by T. Kuphaldt:

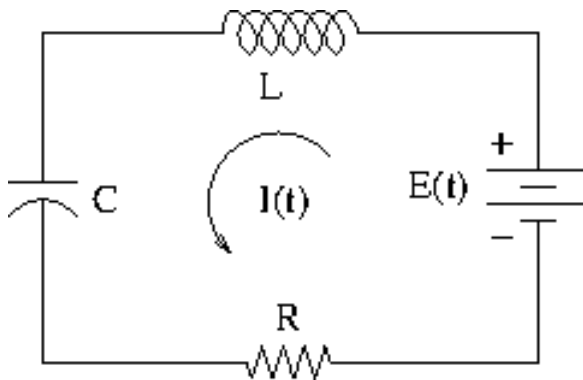
<http://www.allaboutcircuits.com/>.

Course slides on Electric Circuits:

<http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/electricalCircuits.pdf>.

Solved examples of electrical networks can be found in the lecture notes of Ruye Wang:

<http://fourier.eng.hmc.edu/e84/lectures/ch2/node2.html>.

Sample Quiz 6, Problem 3. RLC-Circuit

The Problem. Suppose $E = 100 \sin(20t)$, $L = 5$ H, $R = 250 \Omega$ and $C = 0.002$ F. The model for the charge $Q(t)$ is $LQ'' + RQ' + \frac{1}{C}Q = E(t)$.

- (a) Differentiate the charge model and substitute $I = \frac{dQ}{dt}$ to obtain the current model $5I'' + 250I' + 500I = 2000 \cos(20t)$.
- (b) Find the **reactance** $S = \omega L - \frac{1}{\omega C}$, where $\omega = 20$ is the input frequency, the natural frequency of $E = 100 \sin(20t)$ and $E' = 2000 \cos(20t)$.
- (c) Substitute $I = A \cos(20t) + B \sin(20t)$ into the current model (a) and solve for $A = \frac{-12}{109}$, $B = \frac{40}{109}$. Then the steady-state current is

$$I(t) = A \cos(20t) + B \sin(20t) = \frac{-12 \cos(20t) + 40 \sin(20t)}{109}.$$

- (d) Write the answer in (c) in phase-amplitude form $I = I_0 \sin(20t - \delta)$ with $I_0 > 0$ and $\delta \geq 0$. Then compute the **time lag** δ/ω .

Answers: $I_0 = \frac{4}{\sqrt{109}}$, $\delta = \arctan(3/10)$, $\delta/\omega = 0.01457$.

References

Course slides on Electric Circuits:

<http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/electricalCircuits.pdf>.

Edwards-Penney *Differential Equations and Boundary Value Problems*, sections 3.4, 3.5, 3.6, 3.7.

Solutions to Problem 3

Problem 1(a) Start with $5Q'' + 250Q' + 500Q = 100 \sin(20t)$. Differentiate across to get $5Q''' + 250Q'' + 500Q' = 2000 \cos(20t)$. Change Q' to I .

Problem 1(b) $S = (20)(5) - 1/(20 * 0.002) = 75$

Problem 1(c) It helps to use the differential equation $u'' + 400u = 0$ satisfied by both $u_1 = \cos(20t)$ and $u_2 = \sin(20t)$. Functions u_1, u_2 are Euler solution atoms, hence independent. Along the solution path, we'll use $u_1' = -20 \sin(20t) = -20u_2$ and $u_2' = 20 \cos(20t) = 20u_1$. The arithmetic is simplified by dividing the equation first by 5. We then substitute $I = Au_1 + Bu_2$.

$$\begin{aligned} I'' + 50I' + 100I &= 400 \sin(20t) \\ A(u_1'' + 50u_1' + 100u_1) + B(u_2'' + 50u_2' + 100u_2) &= 400 \sin(20t) \\ A(-400u_1 + 50(-20u_2) + 100u_1) + B(-400u_2 + 50(20u_1) + 100u_2) &= 400 \sin(20t) \\ (-400A + 100A + 1000B)u_1 + (-1000A - 400B + 100B)u_2 &= 400u_2 \end{aligned}$$

By independence of u_1, u_2 , coefficients of u_1, u_2 on each side of the equation must match. The linear algebra property is called *unique representation of linear combinations*. This implies the 2×2 system of equations

$$\begin{aligned} -300A + 1000B &= 0, \\ -1000A - 300B &= 400. \end{aligned}$$

The solution by Cramer's rule (the easiest method) is $A = -12/109, B = 40/109$. Then the steady-state current is

$$I(t) = A \cos(20t) + B \sin(20t) = \frac{-12 \cos(20t) + 40 \sin(20t)}{109}.$$

The **steady-state current** is defined to be the sum of those terms in the general solution of the differential equation that remain after all terms that limit to zero at $t = \infty$ have been removed. The logic is that only these terms contribute to a graphic or to a numerical calculation after enough time has passed (as $t \rightarrow \infty$).

Problem 1(d) Let $\cos(\delta) = B/I_0, \sin(\delta) = -A/I_0, I_0 = \sqrt{A^2 + B^2}$. Use the trig identity

$$\sin(a - b) = \sin(a) \cos(b) - \cos(a) \sin(b)$$

to rearrange the current formula as follows:

$$I(t) = A \cos(20t) + B \sin(20t) = I_0(\sin(20t) \cos(\delta) - \sin(\delta) \cos(20t)) = I_0 \sin(20t - \delta).$$

Compute $I_0 = \sqrt{A^2 + B^2} = \frac{4}{\sqrt{109}}$. Compute $\tan(\delta) = \frac{\sin \delta}{\cos \delta} = -A/B = 12/40$. Then $\delta = \arctan(12/40)$ and finally $\delta/\omega = \arctan(3/10)/20 = 0.01457$.

Another method, using Edwards-Penney Section 3.7: Compute the **impedance** $Z = \sqrt{R^2 + S^2} = \sqrt{250^2 + 75^2} = \sqrt{68125} = 25\sqrt{109}$ and then $I_0 = E_0/Z = 4/\sqrt{109}$. The phase $\delta = \arctan(S/R) = \arctan(75/250) = \arctan(3/10)$. Then the time lag is $\delta/\omega = \frac{\arctan(0.3)}{20} = 0.01457$.

Sample Quiz 6
Solutions

Problem 1. Resistive Network with 2 Loops and DC Sources

(a) The system in augmented matrix form is

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 4 & 2 & 0 & 28 \\ 0 & 2 & -1 & 7 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 6 & 4 & 28 \\ 0 & 2 & -1 & 7 \end{array} \right) \text{ Combo}(1,2,-4)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 7 & 7 \\ 0 & 2 & -1 & 7 \end{array} \right) \text{ Combo}(3,2,-3)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & -1 & 7 \\ 0 & 0 & 7 & 7 \end{array} \right) \text{ Swap}(2,3)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & -1 & 7 \\ 0 & 0 & 1 & 1 \end{array} \right) \text{ mult}(3, 1/7)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & 0 & 8 \\ 0 & 0 & 1 & 1 \end{array} \right) \text{ Combo}(3,2,1)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 2 & 0 & 8 \\ 0 & 0 & 1 & 1 \end{array} \right) \text{ Combo}(3,1,1)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array} \right) \text{ mult}(2, 1/2)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array} \right) \text{ Combo}(2,1,1) \text{ Last frame test passed } \checkmark$$

Sample Quiz 6, Solutions

2

Solution:
$$\begin{cases} I_1 = 5 \\ I_2 = 4 \\ I_3 = 1 \end{cases}$$
 Unique solution case

(b) The voltage drop across a resistor is given by Ohm's Law: $V_R = RI$

Drop across $R_1 = 4 \text{ } \Omega$: $V_{R_1} = R_1 I_1 = 20$

Drop across $R_2 = 2 \text{ } \Omega$: $V_{R_2} = R_2 I_2 = 8$

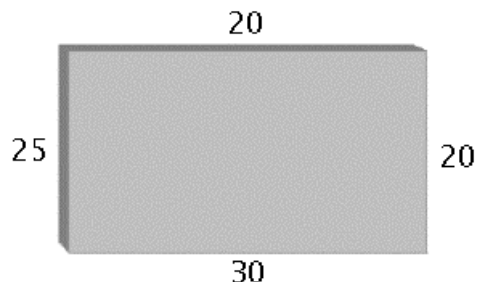
Drop across $R_3 = 1 \text{ } \Omega$: $V_{R_3} = R_3 I_3 = 1$

Sample Quiz 7

Sample Quiz 7, Problem 1. XXXXX
The Problem.

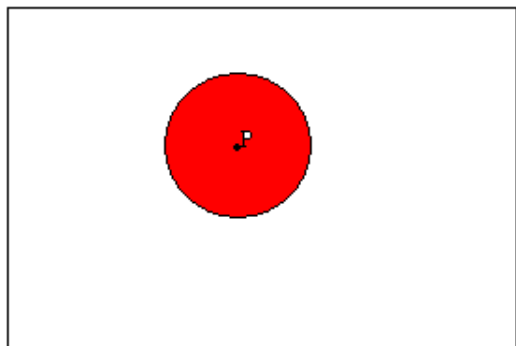
Sample Quiz 7, Problem 3. Heat Transfer and the Mean Value Property.

Consider the cross section of a long rectangular dam on a river, represented in the figure.



The boundaries of the dam are subject to three factors: the temperature in degrees Celsius of the air (20), the water (25), and the ground at its base (30).

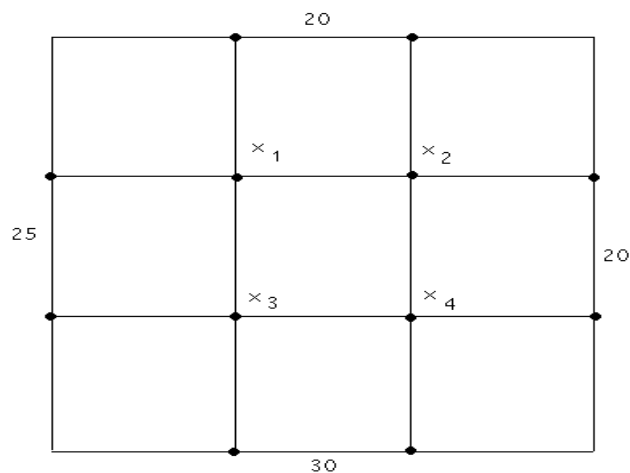
An analysis of the heat transfer from the three sources will be done from the equilibrium temperature, which is found by the Mean Value Property below.



The Mean Value Property

If a plate is at thermal equilibrium, and C is a circle contained in the plate with center P , then the temperature at P is the average value of the temperature function over C .

A version of the Mean Value Property says that the temperature at center P of circle C is the average of the temperatures at four equally-spaced points on C . We construct a grid as in the figure below, label the unknown temperatures at interior grid points as x_1, x_2, x_3, x_4 , then use the property to obtain four equations.



Four-Point Temperature Averages

$$x_1 = \frac{1}{4}(20 + 25 + x_2 + x_3)$$

$$x_2 = \frac{1}{4}(20 + 20 + x_1 + x_4)$$

$$x_3 = \frac{1}{4}(25 + 30 + x_1 + x_4)$$

$$x_4 = \frac{1}{4}(20 + 30 + x_2 + x_3)$$

Solve the equations for the four temperatures $x_1 = 23.125, x_2 = 21.875, x_3 = 25.625, x_4 = 24.375$ by any method.

References. EPH Chapters 12, 13, on heat transfer. Used in Partial Differential Equations 3150. Intro Differential Equations 2280 uses Chapter 9 of a different Edwards-Penney textbook.

Sample Quiz 7 Solutions

Extra Credit

Problem 3. write the equations as

$$\begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 45 \\ 40 \\ 55 \\ 50 \end{pmatrix}$$

The vector \vec{x} is the vector of equilibrium temperatures, located at the grid points of the diagram.

We do swap, combo, mult on the augmented matrix

$$\left(\begin{array}{cccc|c} 4 & -1 & -1 & 0 & 45 \\ -1 & 4 & 0 & -1 & 40 \\ -1 & 0 & 4 & -1 & 55 \\ 0 & -1 & -1 & 4 & 50 \end{array} \right)$$

or alternatively compute the inverse of the coefficient matrix A :

$$A^{-1} = \begin{pmatrix} 7 & 2 & 2 & 1 \\ 2 & 7 & 1 & 2 \\ 2 & 1 & 7 & 2 \\ 1 & 2 & 2 & 7 \end{pmatrix} \cdot \frac{1}{24}$$

The solution of $A\vec{x} = \vec{b}$ is always $\vec{x} = A^{-1}\vec{b}$, so

$$\vec{x} = \frac{1}{24} \begin{pmatrix} 7 & 2 & 2 & 1 \\ 2 & 7 & 1 & 2 \\ 2 & 1 & 7 & 2 \\ 1 & 2 & 2 & 7 \end{pmatrix} \begin{pmatrix} 45 \\ 40 \\ 55 \\ 50 \end{pmatrix} = \begin{pmatrix} 23.125 \\ 21.875 \\ 25.625 \\ 24.375 \end{pmatrix} = \begin{pmatrix} 185 \\ 175 \\ 205 \\ 195 \end{pmatrix} \cdot \frac{1}{8}$$

Most of this can be done with technology. As the grid size goes up, so does the difficulty of writing A, \vec{b} .

Sample Quiz 8

Sample Quiz 8, Problem 1. Solving Higher Order Constant-Coefficient Equations

The **Algorithm** applies to constant-coefficient homogeneous linear differential equations of order N , for example equations like

$$y'' + 16y = 0, \quad y'''' + 4y'' = 0, \quad \frac{d^5 y}{dx^5} + 2y''' + y'' = 0.$$

1. Find the N th degree characteristic equation by Euler's substitution $y = e^{rx}$. For instance, $y'' + 16y = 0$ has characteristic equation $r^2 + 16 = 0$, a polynomial equation of degree $N = 2$.
2. Find all real roots and all complex conjugate pairs of roots satisfying the characteristic equation. List the N roots according to multiplicity.
3. Construct N distinct Euler solution atoms from the list of roots. Then the general solution of the differential equation is a linear combination of the Euler solution atoms with arbitrary coefficients c_1, c_2, c_3, \dots .

The solution space S of the differential equation is given by

$$S = \text{span}(\text{the } N \text{ Euler solution atoms}).$$

Examples: Constructing Euler Solution Atoms from roots.

Three roots $0, 0, 0$ produce three atoms $e^{0x}, xe^{0x}, x^2e^{0x}$ or $1, x, x^2$.

Three roots $0, 0, 2$ produce three atoms e^{0x}, xe^{0x}, e^{2x} .

Two complex conjugate roots $2 \pm 3i$ produce two atoms $e^{2x} \cos(3x), e^{2x} \sin(3x)$.¹

Four complex conjugate roots listed according to multiplicity as $2 \pm 3i, 2 \pm 3i$ produce four atoms $e^{2x} \cos(3x), e^{2x} \sin(3x), xe^{2x} \cos(3x), xe^{2x} \sin(3x)$.

Seven roots $1, 1, 3, 3, 3, \pm 3i$ produce seven atoms $e^x, xe^x, e^{3x}, xe^{3x}, x^2e^{3x}, \cos(3x), \sin(3x)$.

Two conjugate complex roots $a \pm bi$ ($b > 0$) arising from roots of $(r - a)^2 + b^2 = 0$ produce two atoms $e^{ax} \cos(bx), e^{ax} \sin(bx)$.

The Problem

Solve for the general solution or the particular solution satisfying initial conditions.

- (a) $y'' + 16y' = 0$
- (b) $y'' + 16y = 0$
- (c) $y'''' + 16y'' = 0$
- (d) $y'' + 16y = 0, y(0) = 1, y'(0) = -1$
- (e) $y'''' + 9y'' = 0, y(0) = y'(0) = 0, y''(0) = y'''(0) = 1$
- (f) The characteristic equation is $(r - 2)^2(r^2 - 4) = 0$.
- (g) The characteristic equation is $(r - 1)^2(r^2 - 1)((r + 2)^2 + 4) = 0$.
- (h) The characteristic equation roots, listed according to multiplicity, are $0, 0, 0, -1, 2, 2, 3 + 4i, 3 - 4i$.

¹The Reason: $\cos(3x) = \frac{1}{2}e^{3xi} + \frac{1}{2}e^{-3xi}$ by Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$. Then $e^{2x} \cos(3x) = \frac{1}{2}e^{2x+3xi} + \frac{1}{2}e^{2x-3xi}$ is a linear combination of exponentials e^{rx} where r is a root of the characteristic equation. Euler's substitution implies e^{rx} is a solution, so by superposition, so also is $e^{2x} \cos(3x)$. Similar for $e^{2x} \sin(3x)$.

Solutions to Problem 1

(a) $y'' + 16y' = 0$ upon substitution of $y = e^{rx}$ becomes $(r^2 + 16r)e^{rx} = 0$. Cancel e^{rx} to find the **characteristic equation** $r^2 + 16r = 0$. It factors into $r(r + 16) = 0$, then the two roots r make the list $r = 0, -16$. The Euler solution atoms for these roots are e^{0x}, e^{-16x} . Report the general solution $y = c_1 e^{0x} + c_2 e^{-16x} = c_1 + c_2 e^{-16x}$, where symbols c_1, c_2 stand for arbitrary constants.

(b) $y'' + 16y = 0$ has characteristic equation $r^2 + 16 = 0$. Because a quadratic equation $(r - a)^2 + b^2 = 0$ has roots $r = a \pm bi$, then the root list for $r^2 + 16 = 0$ is $0 + 4i, 0 - 4i$, or briefly $\pm 4i$. The Euler solution atoms are $e^{0x} \cos(4x), e^{0x} \sin(4x)$. The general solution is $y = c_1 \cos(4x) + c_2 \sin(4x)$, because $e^{0x} = 1$.

(c) $y'''' + 16y'' = 0$ has characteristic equation $r^4 + 4r^2 = 0$ which factors into $r^2(r^2 + 16) = 0$ having root list $0, 0, 0 \pm 4i$. The Euler solution atoms are $e^{0x}, xe^{0x}, e^{0x} \cos(4x), e^{0x} \sin(4x)$. Then the general solution is $y = c_1 + c_2 x + c_3 \cos(4x) + c_4 \sin(4x)$.

(d) $y'' + 16y = 0, y(0) = 1, y'(0) = -1$ defines a particular solution y . The usual arbitrary constants c_1, c_2 are determined by the initial conditions. From part (b), $y = c_1 \cos(4x) + c_2 \sin(4x)$. Then $y' = -4c_1 \sin(4x) + 4c_2 \cos(4x)$. Initial conditions $y(0) = 1, y'(0) = -1$ imply the equations $c_1 \cos(0) + c_2 \sin(0) = 1, -4c_1 \sin(0) + 4c_2 \cos(0) = -1$. Using $\cos(0) = 1$ and $\sin(0) = 0$ simplifies the equations to $c_1 = 1$ and $4c_2 = -1$. Then the particular solution is $y = c_1 \cos(4x) + c_2 \sin(4x) = \cos(4x) - \frac{1}{4} \sin(4x)$.

(e) $y'''' + 9y'' = 0, y(0) = y'(0) = 0, y''(0) = y'''(0) = 1$ is solved like part (d). First, the characteristic equation $r^4 + 9r^2 = 0$ is factored into $r^2(r^2 + 9) = 0$ to find the root list $0, 0, 0 \pm 3i$. The Euler solution atoms are $e^{0x}, xe^{0x}, e^{0x} \cos(3x), e^{0x} \sin(3x)$, which implies the general solution $y = c_1 + c_2 x + c_3 \cos(3x) + c_4 \sin(3x)$. We have to find the derivatives of y : $y' = c_2 - 3c_3 \sin(3x) + 3c_4 \cos(3x), y'' = -9c_3 \cos(3x) - 9c_4 \sin(3x), y''' = 27c_3 \sin(3x) - 27c_4 \cos(3x)$. The initial conditions give four equations in four unknowns c_1, c_2, c_3, c_4 :

$$\begin{array}{ccccccccc} c_1 & + & c_2(0) & + & c_3 \cos(0) & + & c_4 \sin(0) & = & 0, \\ & & c_2 & - & 3c_3 \sin(0) & + & 3c_4 \cos(0) & = & 0, \\ & & & - & 9c_3 \cos(0) & - & 9c_4 \sin(0) & = & 1, \\ & & & & 27c_3 \sin(0) & - & 27c_4 \cos(0) & = & 1, \end{array}$$

which has invertible coefficient matrix $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -9 & 0 \\ 0 & 0 & 0 & -27 \end{pmatrix}$ and right side vector $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$. The

solution is $c_1 = c_2 = 1/9, c_3 = -1/9, c_4 = -1/27$. Then the particular solution is $y = c_1 + c_2 x + c_3 \cos(3x) + c_4 \sin(3x) = \frac{1}{9} + \frac{1}{9}x - \frac{1}{9} \cos(3x) - \frac{1}{27} \sin(3x)$

(f) The characteristic equation is $(r - 2)^2(r^2 - 4) = 0$. Then $(r - 2)^3(r + 2) = 0$ with root list $2, 2, 2, -2$ and Euler atoms $e^{2x}, xe^{2x}, x^2 e^{2x}, e^{-2x}$. The general solution is a linear combination of these four atoms.

(g) The characteristic equation is $(r - 1)^2(r^2 - 1)((r + 2)^2 + 4) = 0$. The root list is $1, 1, 1, -1, -2 \pm 2i$ with Euler atoms $e^x, xe^x, x^2 e^x, e^{-x}, e^{-2x} \cos(2x), e^{-2x} \sin(2x)$. The general solution is a linear combination of these six atoms.

(h) The characteristic equation roots, listed according to multiplicity, are $0, 0, 0, -1, 2, 2, 3 + 4i, 3 - 4i$. Then the Euler solution atoms are $e^{0x}, xe^{0x}, x^2 e^{0x}, e^{-x}, e^{2x}, xe^{2x}, e^{3x} \cos(4x), e^{3x} \sin(4x)$. The general solution is a linear combination of these eight atoms.

Sample Quiz 8, Problem 2. Laplace Theory

Laplace theory implements the *method of quadrature* for higher order differential equations, linear systems of differential equations, and certain partial differential equations.

Laplace's method solves differential equations.

The Problem. Solve by table methods or Laplace's method.

- (a) Forward table. Find $\mathcal{L}(f(t))$ for $f(t) = te^{2t} + 2t \sin(3t) + 3e^{-t} \cos(4t)$.
 (b) Backward table. Find $f(t)$ for

$$\mathcal{L}(f(t)) = \frac{16}{s^2 + 4} + \frac{s + 1}{s^2 - 2s + 10} + \frac{2}{s^2 + 16}.$$

- (c) Solve the initial value problem $x''(t) + 256x(t) = 1$, $x(0) = 1$, $x'(0) = 0$.

Solution (a).

$$\begin{aligned} \mathcal{L}(f(t)) &= \mathcal{L}(te^{2t} + 2t \sin(3t) + 3e^{-t} \cos(4t)) \\ &= \mathcal{L}(te^{2t}) + 2\mathcal{L}(t \sin(3t)) + 3\mathcal{L}(e^{-t} \cos(4t)) && \text{Linearity} \\ &= -\frac{d}{ds}\mathcal{L}(e^{2t}) - 2\frac{d}{ds}\mathcal{L}(\sin(3t)) + 3\mathcal{L}(e^{-t} \cos(4t)) && \text{Differentiation rule} \\ &= -\frac{d}{ds}\mathcal{L}(e^{2t}) - 2\frac{d}{ds}\mathcal{L}(\sin(3t)) + 3\mathcal{L}(\cos(4t))\Big|_{s=s+1} && \text{Shift rule} \\ &= -\frac{d}{ds}\frac{1}{s-2} - 2\frac{d}{ds}\frac{3}{s^2+9} + 3\frac{s}{s^2+16}\Big|_{s=s+1} && \text{Forward table} \\ &= \frac{1}{(s-2)^2} + \frac{12s}{(s^2+9)^2} + 3\frac{s+1}{(s+1)^2+16} && \text{Calculus} \end{aligned}$$

Solution (b).

$$\begin{aligned} \mathcal{L}(f(t)) &= \frac{16}{s^2+4} + \frac{s+1}{s^2-2s+10} + \frac{2}{s^2+16} \\ &= 8\frac{2}{s^2+4} + \frac{s+1}{(s-1)^2+9} + \frac{1}{2}\frac{4}{s^2+16} && \text{Prep for backward table} \\ &= 8\mathcal{L}(\sin 2t) + \frac{s+1}{(s-1)^2+9} + \frac{1}{2}\mathcal{L}(\sin 4t) && \text{backward table} \\ &= 8\mathcal{L}(\sin 2t) + \frac{s+2}{s^2+9}\Big|_{s=s-1} + \frac{1}{2}\mathcal{L}(\sin 4t) && \text{shift rule} \\ &= 8\mathcal{L}(\sin 2t) + \mathcal{L}(\cos 3t + \frac{2}{3}\sin 3t)\Big|_{s=s-1} + \frac{1}{2}\mathcal{L}(\sin 4t) && \text{backward table} \\ &= 8\mathcal{L}(\sin 2t) + \mathcal{L}(e^t \cos 3t + e^t \frac{2}{3}\sin 3t) + \frac{1}{2}\mathcal{L}(\sin 4t) && \text{shift rule} \\ &= \mathcal{L}(8 \sin 2t) + \mathcal{L}(e^t \cos 3t + e^t \frac{2}{3}\sin 3t + \frac{1}{2}\sin 4t) && \text{Linearity} \\ f(t) &= 8 \sin 2t + e^t \cos 3t + e^t \frac{2}{3}\sin 3t + \frac{1}{2}\sin 4t && \text{Lerch's cancel rule} \end{aligned}$$

Solution (c).

$$\begin{aligned} \mathcal{L}(x''(t) + 256x(t)) &= \mathcal{L}(1) && \mathcal{L} \text{ acts like matrix mult} \\ s\mathcal{L}(x') - x'(0) + 256\mathcal{L}(x) &= \mathcal{L}(1) && \text{Parts rule} \\ s(s\mathcal{L}(x) - x(0)) - x'(0) + 256\mathcal{L}(x) &= \mathcal{L}(1) && \text{Parts rule} \\ s^2\mathcal{L}(x) - s + 256\mathcal{L}(x) &= \mathcal{L}(1) && \text{Use } x(0) = 1, x'(0) = 0 \\ (s^2 + 256)\mathcal{L}(x) &= s + \mathcal{L}(1) && \text{Collect } \mathcal{L}(x) \text{ left} \end{aligned}$$

$$\begin{aligned} \mathcal{L}(x) &= \frac{s+\mathcal{L}(1)}{(s^2+256)} && \text{Isolate } \mathcal{L}(x) \text{ left} \\ \mathcal{L}(x) &= \frac{s+1/s}{(s^2+256)} && \text{Forward table} \\ \mathcal{L}(x) &= \frac{s^2+1}{s(s^2+256)} && \text{Algebra} \\ \mathcal{L}(x) &= \frac{A}{s} + \frac{Bs+C}{s^2+256} && \text{Partial fractions} \\ \mathcal{L}(x) &= A\mathcal{L}(1) + B\mathcal{L}(\cos 16t) + \frac{C}{16}\mathcal{L}(\sin 16t) && \text{Backward table} \\ \mathcal{L}(x) &= \mathcal{L}(A + B \cos 16t + \frac{C}{16} \sin 16t) && \text{Linearity} \\ x(t) &= A + B \cos 16t + \frac{C}{16} \sin 16t && \text{Lerch's rule} \end{aligned}$$

The partial fraction problem remains:

$$\frac{s^2 + 1}{s(s^2 + 256)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 256}$$

This problem is solved by clearing the fractions, then swapping sides of the equation, to obtain

$$A(s^2 + 256) + (Bs + C)(s) = s^2 + 1.$$

Substitute three values for s to find 3 equations in 3 unknowns A, B, C :

$$\begin{aligned} s = 0 & \quad 256A & = & 1 \\ s = 1 & \quad 257A + B + C & = & 2 \\ s = -1 & \quad 257A + B - C & = & 2 \end{aligned}$$

Then $A = 1/256, B = 255/256, C = 0$ and finally

$$x(t) = A + B \cos 16t + \frac{C}{16} \sin 16t = \frac{1 + 255 \cos 16t}{256}$$

Answer Checks

```
# Sample quiz 8
# answer check problem 2(a)
f:=t*exp(2*t)+2*t*sin(3*t)+3*exp(-t)*cos(4*t);
with(inttrans): # load laplace package
laplace(f,t,s);
# The last two fractions simplify to 3(s+1)/((s+1)^2+16).
# answer check problem 2(b)
F:=16/(s^2+4)+(s+1)/(s^2-2*s+10)+2/(s^2+16);
invlaplace(F,s,t);
# answer check problem 2(c)
de:=diff(x(t),t,t)+256*x(t)=1;ic:=x(0)=1,D(x)(0)=0;
dsolve([de,ic],x(t));
# answer check problem 2(c), partial fractions
convert((s^2+1)/(s*(s^2+256)),parfrac,s);
```

The output appears on the next page

```

-> # Sample quiz 11
-> # answer check problem 2(a)
-> f:=t*exp(2*t)+2*t*sin(3*t)+3*exp(-t)*cos(4*t);
      f:= t e2t + 2 t sin(3 t) + 3 e-t cos(4 t) (1)
-> with(inttrans): # load laplace package
-> laplace(f,t,s) assuming s::real;
      
$$\frac{1}{(s-2)^2} + \frac{12s}{(s^2+9)^2} + \frac{3}{2(s+1-4I)} + \frac{3}{2(s+1+4I)}$$
 (2)
-> # The last two fractions simplify to 3(s+1)/((s+1)^2+16).
-> # answer check problem 2(b)
-> F:=16/(s^2+4)+(s+1)/(s^2-2*s+10)+2/(s^2+16);
      F:=  $\frac{16}{s^2+4} + \frac{s+1}{s^2-2s+10} + \frac{2}{s^2+16}$  (3)
-> invlaplace(F,s,t);
      
$$8 \sin(2 t) + \frac{1}{2} \sin(4 t) + \frac{1}{3} e^t (3 \cos(3 t) + 2 \sin(3 t))$$
 (4)
-> # answer check problem 2(c)
-> de:=diff(x(t),t,t)+256*x(t)=1;ic:=x(0)=1,D(x)(0)=0;
      de:=  $\frac{d^2}{dt^2} x(t) + 256 x(t) = 1$ 
      ic:= x(0) = 1, D(x)(0) = 0 (5)
-> dsolve([de,ic],x(t));
      x(t) =  $\frac{1}{256} + \frac{255}{256} \cos(16 t)$  (6)
-> # answer check problem 2(c), partial fractions
-> convert((s^2+1)/(s*(s^2+256)),parfrac,s);
      
$$\frac{1}{256 s} + \frac{255}{256} \frac{s}{s^2 + 256}$$
 (7)

```

Sample Quiz 9

Extra Credit Problem 1. Piecewise Continuous Inputs

Consider a passenger SUV on a one-day trip from Salt Lake City to Pine Bluffs, Wyoming, on the Nebraska border. The route is I-80 E, 471 miles through Utah and Wyoming. Google maps estimates 6 hours and 52 minutes hours driving time. The table below shows the distances, time, road segment and average speed with total trip time 7 hours and 42 minutes. Cities enroute reduce the freeway speed by 10 mph, the trip time effect not shown in the table.

Miles	Minutes	Speed mph	Road Segment	Posted limit mph
18.1	20	54.3	Parley's Walmart to Kimball	65
11.3	12	56.3	Kimball to Wanship	65 – 55
9.1	13	42	Wanship to Coalville	70
5.7	7	48.9	Coalville to Echo Dam	70
16.5	18	55	Echo Dam to 75 mph sign	70
39	36	65	75 mph sign to Evanston	75
308	269	68.7	Evanston to Laramie	75
50.6	50	61.2	Laramie to Cheyenne	75
43	37	69.7	Cheyenne to Pine bluffs	75

The **velocity function** for the SUV is approximated by

$$V_{pc}(t) = \left\{ \begin{array}{lll} \text{Speed mph} & \text{Time interval minutes} & \text{Road segment} \\ \hline 54.3 & 0 < t < 20 & \text{Parley's Walmart to Kimball} \\ 56.3 & 20 < t < 32 & \text{Kimball to Wanship} \\ 42.0 & 32 < t < 45 & \text{Wanship to Coalville} \\ 48.9 & 45 < t < 52 & \text{Coalville to Echo Dam} \\ 55.0 & 52 < t < 70 & \text{Echo Dam to 75 mph sign} \\ 65.0 & 70 < t < 106 & \text{75 mph sign to Evanston} \\ 68.7 & 106 < t < 375 & \text{Evanston to Laramie} \\ 61.2 & 375 < t < 425 & \text{Laramie to Cheyenne} \\ 69.7 & 425 < t < 462 & \text{Cheyenne to Pine bluffs} \\ 0.00 & 462 < t < \infty & \text{SUV stopped} \end{array} \right.$$

The velocity function $V_{pc}(t)$ is piecewise continuous, because it has the general form

$$f(t) = \left\{ \begin{array}{ll} f_1(t) & t_1 < t < t_2 \\ f_2(t) & t_2 < t < t_3 \\ \vdots & \vdots \\ f_n(t) & t_n < t < t_{n+1} \end{array} \right.$$

where functions f_1, f_2, \dots, f_n are **continuous on the whole real line** $-\infty < t < \infty$. We don't define $f(t)$ at division points, because of many possible ways to make the definition. As long as these values are not used, then it will make no difference. Both right and left hand limits exist at a division point. For Laplace theory, we like the definition $f(t_k) = \lim_{h \rightarrow 0^+} f(t_k + h)$, which makes the function right-continuous.

The Problem. The SUV travels from $t = 0$ to $t = \frac{462}{60} = 7.7$ hours. The odometer trip meter reading $x(t)$ is in miles (assume $x(0) = 0$). The function $V_{pc}(t)$ is an approximation to the speedometer reading. Laplace's method can solve the approximation model

$$\frac{dx}{dt} = V_{pc}(60t), \quad x(0) = 0, \quad x \text{ in miles, } t \text{ in hours,}$$

obtaining $x(t) = \int_0^t V_{\text{pc}}(60w)dw$, the same result as the method of quadrature. Show the details. Then display the piecewise linear continuous trip meter reading $x(t)$.

Solution.

Method of Quadrature. The meaning of the differential equation is that $x'(t)$ is piecewise continuous. We want $x(t)$ to be continuous, because it is the odometer trip meter reading. But $x'(t)$ cannot be continuous, if we require $\frac{dx}{dt} = V_{\text{pc}}(60t)$, because the right side is piecewise defined and discontinuous at division points.

Theorem (Fundamental Theorem of Calculus)

If $f'(x)$ is piecewise continuous and $f(x)$ is continuous on $a \leq x \leq b$, then $\int_a^b f'(x)dx = f(b) - f(a)$.

The theorem implies that the method of quadrature works for the equation $x'(t) = V_{\text{pc}}(60t)$. The quadrature method gives the correct answer

$$x(t) = \int_0^t V_{\text{pc}}(60w)dw.$$

Another plan is to split $x'(t) = V_{\text{pc}}(60t)$ into 10 simple equations, $x' = 54.3$, $x(0) = 0$ on $0 \leq t < 20$ being the first equation. The next equation is $x' = 56.3$, $x(20) = x_0$, on $20 < t < 32$. To make $x(t)$ continuous, we must choose $x_0 = 1086$, which is the value at the division point $t = 20$ assumed by the first problem ($x' = 54.3$, $x(0) = 0$ on $0 \leq t < 20$). This tedious process has to be continued for all 10 segments. The result is that $x(t)$ is **piecewise linear** between division points.

Laplace's Method. The piecewise continuous input $V_{\text{pc}}(60t)$ is of exponential order, because it is zero after $t = 462/60$. Laplace theory says it has a Laplace transform $\mathcal{L}(V_{\text{pc}}(60t))$. Assuming a continuous solution $x(t)$, with $x'(t)$ piecewise continuous, then the equation to be satisfied is

$$s\mathcal{L}(x(t)) - x(0) = \mathcal{L}(x'(t)) = \mathcal{L}(V_{\text{pc}}(60t)).$$

The **Laplace integral theorem** implies

$$\mathcal{L}(x(t)) = \frac{1}{s}\mathcal{L}(V_{\text{pc}}(60t)) = \mathcal{L}\left(\int_0^t V_{\text{pc}}(60w)dw\right).$$

Lerch's theorem then implies that the symbol \mathcal{L} cancel from each side, giving the odometer trip meter reading in terms of the integral of the piecewise continuous input $V_{\text{pc}}(60t)$:

$$x(t) = \int_0^t V_{\text{pc}}(60w)dw.$$

We'll use technology to program and evaluate the integral, even though it can be done by hand. The plan is to plot the trip meter reading, then comment on the slow and fast segments of the route, by using a clever plot involving the average speed. The last display is the piecewise linear trip meter reading $x(t)$.

Maple

```
Xpc:=t->piecewise(t<0,0,
t < 20 ,54.3, t < 32, 56.3, t < 45, 42, t < 52, 48.9,
t < 70 ,55, t < 106, 65, t < 375, 68.7, t < 425, 61.2,
t < 462, 69.7, 0.0);
X:=t->int(Xpc(60*w),w=0..t);
plot(X(t),t=0..480/60); # Almost a straight line.
```

Average Speed

Define the average value of a function $f(w)$ on $a \leq w \leq b$ by $\frac{1}{b-a} \int_a^b f(w)dw$. Then the average speed in the example is

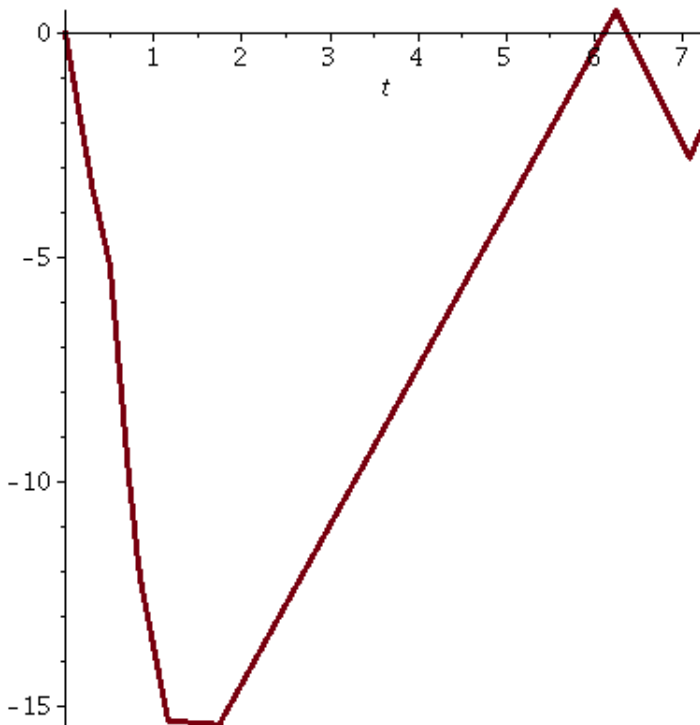
$$\frac{\int_0^{462/60} V_{\text{pc}}(60w)dw}{462/60} = 65.14956710.$$

A Clever Plot

An average driver would try to maintain 65.15 mph. The clever plot will create a graphic of $x(t) - 65.15t$ on interval $0 \leq t \leq T_1$, where T_1 is the 471 mile trip time at 65.15 mph.

Maple code

```
Xpc:=t->piecewise(t<0,0,
t < 20 ,54.3, t < 32, 56.3, t < 45, 42, t < 52, 48.9,
t < 70 ,55, t < 106, 65, t < 375, 68.7, t < 425, 61.2,
t < 462, 69.7, 0.0);
X:=t->int(Xpc(60*w),w=0..t);
AVEspeed:=X(462/60)/(462/60); # AVEspeed = 65.14956710 mph
T1:=solve(AVEspeed*t=471,t); # T1 = 7.229518491 hours
plot(X(t)-AVEspeed*t,t=0..T1);
```



We see from the graphic that segments of the road cause a slowdown of up to 15 mph, but for a brief interval it is possible to exceed the average speed, due to a 75 mph speed limit.

Maple code for piecewise linear display

```
X:=t->int(Xpc(60*w),w=0..t);
convert(X(t),piecewise,t):evalf(%,4);
```

$$\text{Trip meter at time } t = \begin{cases} 0.0 & t \leq 0.0 \\ 54.30t & t \leq 0.3333 \\ 56.30t - 0.6667 & t \leq 0.53 \\ 42.0t + 6.960 & t \leq 0.75 \\ 48.90t + 1.785 & t \leq 0.8667 \\ 55.0t - 3.502 & t \leq 1.167 \\ 65.0t - 15.17 & t \leq 1.767 \\ 68.70t - 21.70 & t \leq 6.25 \\ 61.20t + 25.17 & t \leq 7.083 \\ 69.70t - 35.04 & t \leq 7.7 \\ 501.7 & 7.7 < t \end{cases}$$

Sample Quiz 11

Background. Switches and Impulses

Laplace's method solves differential equations. It is the preferred method for solving equations containing switches or impulses.

Unit Step Define $u(t - a) = \begin{cases} 1 & t \geq a, \\ 0 & t < a. \end{cases}$. It is a **switch**, turned on at $t = a$.

Ramp Define **ramp** $(t - a) = (t - a)u(t - a) = \begin{cases} t - a & t \geq a, \\ 0 & t < a. \end{cases}$, whose graph shape is a continuous **ramp** at 45-degree incline starting at $t = a$.

Unit Pulse Define **pulse** $(t, a, b) = \begin{cases} 1 & a \leq t < b, \\ 0 & \text{otherwise} \end{cases} = u(t - a) - u(t - b)$. The switch is **ON** at time $t = a$ and then **OFF** at time $t = b$.

Impulse of a Force

Define the **impulse** of an applied force $F(t)$ on time interval $a \leq t \leq b$ by the equation

$$\text{Impulse of } F = \int_a^b F(t)dt = \left(\frac{\int_a^b F(t)dt}{b - a} \right) (b - a) = \text{Average Force} \times \text{Duration Time}.$$

Dirac Unit Impulse

A Dirac impulse acts like a hammer hit, a brief injection of energy into a system. It is a special idealization of a real hammer hit, in which only the **impulse** of the force is deemed important, and not its magnitude nor duration.

Define the **Dirac Unit Impulse** by the equation $\delta(t - a) = \frac{du}{dt}(t - a)$, where $u(t - a)$ is the unit step. Symbol δ makes sense only under an integral sign, and the integral in question must be a generalized Riemann integral (definition pending), with new evaluation rules. Symbol δ is an abbreviation like **etc** or **e.g.**, because it abbreviates a paragraph of descriptive text.

- Symbol $M\delta(t - a)$ represents an ideal impulse of magnitude M at time $t = a$. Value M is the change in momentum, but $M\delta(t - a)$ contains no detail about the applied force or the duration. A common force approximation for a hammer hit of very small duration $2h$ and impulse M is Dirac's approximation

$$F_h(t) = \frac{M}{2h} \text{pulse}(t, a - h, a + h).$$

- Symbol δ is not manipulated as an ordinary function. It is a special modeling tool with rules for application and rules for algebraic manipulation.

THEOREM (Second Shifting Theorem). Let $f(t)$ and $g(t)$ be piecewise continuous and of exponential order. Then for $a \geq 0$,

$$\begin{aligned} e^{-as} \mathcal{L}(f(t)) &= \mathcal{L}(f(t)u(t)|_{t:=t-a}), \\ \mathcal{L}(g(t)u(t - a)) &= e^{-as} \mathcal{L}(g(t)|_{t:=t+a}). \end{aligned}$$

Problem 1. Solve the following by Laplace methods.

(a) Forward table. Compute the Laplace integral for the unit step, ramp and pulse, in these special cases:

$$(1) \mathcal{L}(10u(t - \pi)) \quad (2) \mathcal{L}(\mathbf{ramp}(t - 2\pi)), \quad (3) \mathcal{L}(10 \mathbf{pulse}(t, 3, 5)).$$

(b) Backward table. Find $f(t)$ in the following special cases.

$$(1) \mathcal{L}(f) = \frac{5e^{-3s}}{s} \quad (2) \mathcal{L}(f) = \frac{e^{-4s}}{s^2} \quad (3) \mathcal{L}(f) = \frac{5}{s} (e^{-2s} - e^{-3s}).$$

Problem 2. Solve the following Dirac impulse problems.

(c) Dirac Impulse and the Second Shifting theorem. Solve the following forward table problems.

$$(1) \mathcal{L}(10\delta(t - \pi)), \quad (2) \mathcal{L}(5\delta(t - 1) + 10\delta(t - 2) + 15\delta(t - 3)), \quad (2) \mathcal{L}((t - \pi)\delta(t - \pi)).$$

The sum of Dirac impulses in (2) is called an **impulse train**.

Solutions

Solution (a). The forward second shifting theorem applies.

$$(1) \quad \mathcal{L}(10u(t - \pi)) = \mathcal{L}(g(t)u(t - a)) \text{ where } g(t) = 10 \text{ and } a = \pi. \text{ Then } \mathcal{L}(10u(t - \pi)) = \mathcal{L}(g(t)u(t - a)) = e^{-as} \mathcal{L}(g(t)|_{t=t+a}) = e^{-\pi s} \mathcal{L}(10|_{t=t+\pi}) = \frac{10}{s} e^{-\pi s}.$$

$$(2) \quad \mathcal{L}(\mathbf{ramp}(t - 2\pi)) = \mathcal{L}((t - 2\pi)u(t - 2\pi)) = \mathcal{L}(tu(t)|_{t=t-2\pi}) = e^{-2\pi s} \mathcal{L}(t) = \frac{1}{s^2} e^{-2\pi s}.$$

$$(3) \quad \mathcal{L}(10 \mathbf{pulse}(t, 3, 5)) = 10 \mathcal{L}(u(t - 3) - u(t - 5)) = \frac{10}{s}(e^{3s} - e^{5s}).$$

Solution (b). Presence of an exponential e^{-as} signals step $u(t - a)$ in the answer, the main tool being the backward second shifting theorem.

$$(1) \quad \mathcal{L}(f) = \frac{5e^{-3s}}{s} = e^{-3s} \frac{5}{s} = e^{-3s} \mathcal{L}(5) = \mathcal{L}(5u(t)|_{t=t+3}) = \mathcal{L}(5u(t - 3)). \text{ Lerch implies } f = 5u(t - 3).$$

$$(2) \quad \mathcal{L}(f) = \frac{e^{-4s}}{s^2} = \frac{e^{-as}}{s^2} \text{ where } a = 4. \text{ Then } \mathcal{L}(f) = \frac{e^{-as}}{s^2} = \mathcal{L}(tu(t)|_{t=t-a}) = \mathcal{L}((t - 4)u(t - 4)) = \mathcal{L}(\mathbf{ramp}(t - 4)). \text{ Lerch implies } f = \mathbf{ramp}(t - 4).$$

$$(3) \quad \mathcal{L}(f) = e^{-2s} \frac{5}{s} - e^{-3s} \frac{5}{s} = \mathcal{L}(5u(t - 2)) - \mathcal{L}(5u(t - 3)) = \mathcal{L}(5 \mathbf{pulse}(t, 2, 3)). \text{ Lerch implies } f = 5 \mathbf{pulse}(t, 2, 3).$$

Solution (c). The main result for Dirac unit impulse δ is the equation

$$\int_0^{\infty} g(t)\delta(t - a)dt = g(a),$$

valid for $g(t)$ continuous on $0 \leq t < \infty$. When $g(t) = e^{-st}$, then the equation implies the Laplace formula $\mathcal{L}(\delta(t - a)) = e^{-as}$.

$$(1) \quad \mathcal{L}(10\delta(t - \pi)) = 10e^{-\pi s}, \text{ by the displayed equation with } g(t) = 10e^{-st}, \text{ or by using linearity and the formula } \mathcal{L}(\delta(t - a)) = e^{-as}.$$

$$(2) \quad \mathcal{L}(5\delta(t - 1) + 10\delta(t - 2) + 15\delta(t - 3)) = 5 \mathcal{L}(\delta(t - 1)) + 10 \mathcal{L}(\delta(t - 2)) + 15 \mathcal{L}(\delta(t - 3)) = 5e^{-s} + 10e^{-2s} + 15e^{-3s}.$$

$$(3) \quad \mathcal{L}((t - \pi)\delta(t - 2\pi)) = \int_0^{\infty} (t - \pi)e^{st}\delta(t - 2\pi)dt = (t - \pi)e^{-st}|_{t=2\pi} = \pi e^{-2\pi s}, \text{ using } g(t) = (t - \pi)e^{-st} \text{ and } a = 2\pi \text{ in the equation.}$$

Problem 3. Experiment to Find the Transfer Function $h(t)$

Consider a second order problem

$$ax''(t) + bx'(t) + cx(t) = f(t)$$

which by Laplace theory has a particular solution defined as the convolution of the transfer function $h(t)$ with the input $f(t)$,

$$x_p(t) = \int_0^t f(w)h(t-w)dw.$$

Examined in this problem is another way to find $h(t)$, which is the system response to a Dirac unit impulse with zero data. Then $h(t)$ is the solution of

$$ah''(t) + bh'(t) + ch(t) = \delta(t), \quad h(0) = h'(0) = 0.$$

The Problem. Assume a, b, c are constants and define $g(t) = \int_0^t h(w)dw$.

(a) Show that $g(0) = g'(0) = 0$, which means g has zero data.

(b) Let $u(t)$ be the unit step. Argue that g is the solution of

$$ag''(t) + bg'(t) + cg(t) = u(t), \quad g(0) = g'(0) = 0.$$

The fundamental theorem of calculus says that $h(t) = g'(t)$. Therefore, to compute the transfer function $h(t)$, find the response $g(t)$ to the unit step with zero data, followed by computing the derivative $g'(t)$, which equals $h(t)$.

The experimental impact is important. Turning on a switch creates a unit step, generally easier than designing a hammer hit.

(c) Illustrate the method for finding the transfer function $h(t)$ in the special case

$$x''(t) + 2x'(t) + 5x(t) = f(t).$$

Solutions

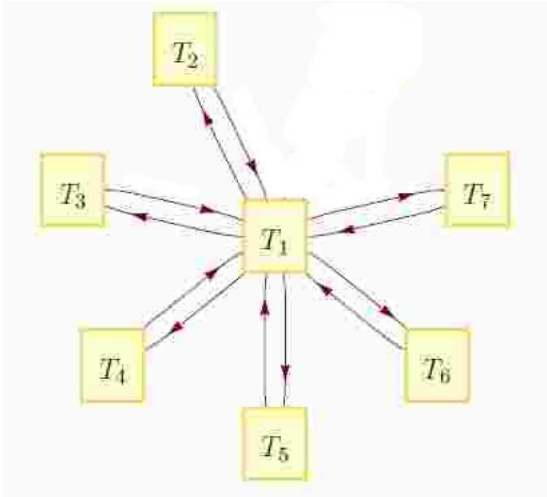
(a) $g(0) = \int_0^0 h(w)dw = 0$, $g'(0) = h'(0) = 0$.

(b) Let $u(t)$ be the unit step. Initial data was decided in part (a). The Laplace applied to $ag''(t) + bg'(t) + cg(t) = u(t)$ gives $(as^2 + bs + c)\mathcal{L}(g) = \mathcal{L}(u(t))$. Then $\mathcal{L}(g) = \mathcal{L}(h(t))\mathcal{L}(u(t)) = \mathcal{L}(h(t))\frac{1}{s}\mathcal{L}\left(\int_0^t h(r)du\right)$ by the **integral theorem**. Lerch's theorem then says $g(t) = \int_0^t h(r)dr$.

(c) For equation $x''(t) + 2x'(t) + 5x(t) = f(t)$ we replace $x(t)$ by $g(t)$ and $f(t)$ by the unit step $u(t)$, then solve $g''(t) + 2g'(t) + 5g(t) = u(t)$, obtaining $\mathcal{L}(g) = \frac{1}{s} \frac{1}{s^2 + 2s + 5} = \mathcal{L}\left(\frac{1}{5} - \frac{1}{10}e^{-t}(2\cos(2t) + \sin(2t))\right)$. Then $g(t) = \frac{1}{5} - \frac{1}{10}e^{-t}(2\cos(2t) + \sin(2t))$ and $h(t) = g'(t) = \frac{1}{2}e^{-t}\sin(2t)$.

Sample Quiz 12

Problem 1.



Flow through each pipe is f gallons per unit time.

Each tank has constant volume V .

Symbols $x_1(t)$ to $x_7(t)$ are the salt amounts in tanks T_1 to T_7 , respectively.

The differential equations are obtained by the classical balance law, which says that the rate of change in salt amount is the rate in minus the rate out. Individual rates in/out are of the form (flow rate)(salt concentration), where flow rate f has units volume per unit time and $x_i(t)/V$ is the concentration = amount/volume.

$$\begin{aligned}x_1'(t) &= \frac{f}{V} (x_2(t) + x_3(t) + x_4(t) + x_5(t) + x_6(t) + x_7(t) - 6x_1(t)) \\x_2'(t) &= \frac{f}{V} (x_1(t) - x_2(t)), \\x_3'(t) &= \frac{f}{V} (x_1(t) - x_3(t)), \\x_4'(t) &= \frac{f}{V} (x_1(t) - x_4(t)), \\x_5'(t) &= \frac{f}{V} (x_1(t) - x_5(t)), \\x_6'(t) &= \frac{f}{V} (x_1(t) - x_6(t)), \\x_7'(t) &= \frac{f}{V} (x_1(t) - x_7(t)).\end{aligned}$$

Problem 1(a). Change variables $t = Vr/f$ to obtain the new system

$$\begin{aligned}\frac{dx_1}{dr} &= x_2 + x_3 + x_4 + x_5 + x_6 + x_7 - 6x_1 \\ \frac{dx_2}{dr} &= x_1 - x_2, \\ \frac{dx_3}{dr} &= x_1 - x_3, \\ \frac{dx_4}{dr} &= x_1 - x_4, \\ \frac{dx_5}{dr} &= x_1 - x_5, \\ \frac{dx_6}{dr} &= x_1 - x_6, \\ \frac{dx_7}{dr} &= x_1 - x_7.\end{aligned}$$

Solution 1(a): Because $\frac{dx(t)}{dt} = \frac{dx}{dr} \frac{dr}{dt} = \frac{dx}{dr} \frac{f}{V}$, then fraction f/V cancels resulting in the new system.

Problem 1(b). Formulate the equations in 1(a) in the system form $\frac{d}{dr}\vec{u} = A\vec{u}$.

Answer:

$$A = \begin{pmatrix} -6 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

Problem 1(c). Find the eigenvalues of A .

Answer: $\lambda = 0, -1, -1, -1, -1, -1, -7$

Solution 1(c).

Let $D = |A - \lambda I|$. Replace $-1 - \lambda$ by symbol u . Then

$$D = \begin{vmatrix} -5 + u & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & u & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & u & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & u & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & u & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & u & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & u \end{vmatrix}$$

Add each of rows 2, 3, 4, 5, 6 to row 1. Then $1 + u$ is a common factor of row 1 and the determinant multiply rule implies

$$D = (1 + u) \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & u & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & u & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & u & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & u & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & u & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & u \end{vmatrix}$$

Cofactor expansion along the last row, plus induction, gives the answer $D = (u + 1)(u - 6)u^5 = (-\lambda)(-\lambda - 7)(-\lambda - 1)^5$ with roots $\lambda = 0, -7, -1, -1, -1, -1, -1$.

Problem 1(d). Find the eigenvectors of A .

Solution 1(d).

The root $\lambda = -1$ causes us to solve $(A + I)\vec{v} = \vec{0}$, which has coefficient matrix

$$B = \begin{pmatrix} -5 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{with} \quad \text{rref}(B) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

There are 2 lead variables and 5 free variables, hence 5 basis vectors

$$\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The eigenvector for $\lambda = 0$ has all components equal to 1. This fact is found from the equation $(A - (0)I)\vec{v} = \vec{0}$, which has coefficient matrix A .

The eigenvector for $\lambda = -7$ has first component -6 and the remaining equal to 1. The task begins with the equation $(A - (-7)I)\vec{v} = \vec{0}$, which has coefficient matrix

$$\begin{pmatrix} 7-6 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 7-1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 7-1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 7-1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 7-1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 7-1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 7-1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 6 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 6 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 6 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 6 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 6 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

The eigenvectors for $\lambda = 0$ and $\lambda = -7$ are respectively

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -6 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Problem 1(e). Solve the differential equation $\frac{d\vec{u}}{dt} = A\vec{u}$ by the eigenanalysis method.

Three Methods for Solving $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$

- **Eigenanalysis Method.** The eigenpairs of matrix A are required. The matrix A must be diagonalizable, meaning there are n eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), \dots, (\lambda_n, \vec{v}_n)$. The main theorem says that the general solution of $\vec{u}' = A\vec{u}$ is

$$\vec{u}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + \dots + c_n e^{\lambda_n t} \vec{v}_n.$$

- **Laplace's Method.** Solve the scalar equations by the Laplace transform method. The **resolvent method** automates this process: $\vec{u}(t) = \mathcal{L}^{-1}((sI - A)^{-1})\vec{u}(0)$.
- **Cayley-Hamilton-Ziebur Method.** The solution $\vec{u}(t)$ is a vector linear combination of the Euler solution atoms f_1, \dots, f_n found from the roots of the characteristic equation $|A - \lambda I| = 0$. The vectors $\vec{d}_1, \dots, \vec{d}_n$ in the linear combination

$$\vec{u}(t) = f_1(t)\vec{d}_1 + f_2(t)\vec{d}_2 + \dots + f_n(t)\vec{d}_n$$

are determined by the explicit formula

$$\langle \vec{d}_1 | \vec{d}_2 | \dots | \vec{d}_n \rangle = \langle \vec{u}_0 | A\vec{u}_0 | \dots | A^{n-1}\vec{u}_0 \rangle \left(W(0)^T\right)^{-1},$$

where $W(t)$ is the Wronskian matrix of atoms f_1, \dots, f_n and \vec{u}_0 is the initial data.

Solution 1(e).

The eigenvectors corresponding to $\lambda = 0, -7, -1, -1, -1, -1, -1$ are respectively

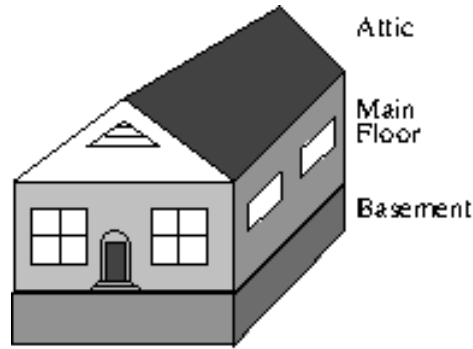
$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -6 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The Eigenanalysis method then implies the solution $\vec{x}(r)$ of $\frac{d\vec{x}}{dr} = A\vec{x}$ is given for arbitrary constants c_1, \dots, c_7 by the expression

$$c_1 e^{0r} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-7r} \begin{pmatrix} -6 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{-r} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_4 e^{-r} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_5 e^{-r} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_6 e^{-r} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_7 e^{-r} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Problem 2. Home Heating

Consider a typical home with attic, basement and insulated main floor.



Heating Assumptions and Variables

- It is usual to surround the main living area with insulation, but the attic area has walls and ceiling without insulation.
- The walls and floor in the basement are insulated by earth.
- The basement ceiling is insulated by air space in the joists, a layer of flooring on the main floor and a layer of drywall in the basement.

The changing temperatures in the three levels is modeled by Newton's cooling law and the variables

$$\begin{aligned}z(t) &= \text{Temperature in the attic,} \\y(t) &= \text{Temperature in the main living area,} \\x(t) &= \text{Temperature in the basement,} \\t &= \text{Time in hours.}\end{aligned}$$

A typical mathematical model is the set of equations

$$\begin{aligned}x' &= \frac{1}{2}(45 - x) + \frac{1}{2}(y - x), \\y' &= \frac{1}{2}(x - y) + \frac{1}{4}(35 - y) + \frac{1}{4}(z - y) + 20, \\z' &= \frac{1}{4}(y - z) + \frac{3}{4}(35 - z).\end{aligned}$$

Problem 2(a). Formulate the system of differential equations as a matrix system $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t) + \vec{b}$. Show details.

Answer. $\vec{u} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$, $\vec{b} = \begin{pmatrix} \frac{45}{2} \\ 20 + \frac{35}{4} \\ \frac{105}{4} \end{pmatrix}$, $A = \begin{pmatrix} -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{4} \\ 0 & \frac{1}{4} & -1 \end{pmatrix}$

Solution Details.

Expand the right side of the system as follows.

$$\begin{aligned}x' &= \frac{45}{2} - \frac{x}{2} + \frac{y}{2} - \frac{x}{2}, \\y' &= \frac{x}{2} - \frac{y}{2} + \frac{35}{4} - \frac{y}{4} + \frac{z}{4} - \frac{y}{4} + 20 \\z' &= \frac{y}{4} - \frac{z}{4} + \frac{105}{4} - \frac{3z}{4}.\end{aligned}$$

Then collect on the variables:

$$\begin{aligned}x' &= -x + \frac{y}{2} + \frac{45}{2}, \\y' &= \frac{x}{2} - y + \frac{z}{4} + 20 + \frac{35}{4} \\z' &= \frac{y}{4} - z + \frac{105}{4}.\end{aligned}$$

The right side of this system can be written as $A\vec{u} + \vec{b}$. Vector \vec{b} is obtained by formally setting $x = y = z = 0$ on the right. This justifies the answer given.

The matrix A has columns equal to the partial derivatives $\partial_x, \partial_y, \partial_z$ of the right side of the scalar system. This idea is important, because it allows the computation of matrix A **without any of the preceding details**.

Problem 2(b). The heating problem has an **equilibrium solution** $\vec{u}_p(t)$ which is a constant vector of temperatures for the three floors. It is formally found by setting $\frac{d}{dt}\vec{u}(t) = 0$, and then $\vec{u}_p = -A^{-1}\vec{b}$. Justify the algebra and explicitly find $\vec{u}_p(t)$.

Answer 2(b). $\vec{u}_p(t) = -A^{-1}\vec{b} = \begin{pmatrix} \frac{620}{11} \\ \frac{745}{11} \\ \frac{475}{11} \end{pmatrix} = \begin{pmatrix} 56.36 \\ 67.73 \\ 43.18 \end{pmatrix}.$

Solution Details.

The equation upon setting the derivative equal to zero becomes $\vec{0} = A\vec{u} + \vec{b}$ which implies $A\vec{u} = -\vec{b}$ and finally $\vec{u} = -A^{-1}\vec{b}$.

The calculation is done by technology. The maple code:

```
A:=<-1,1/2,0|1/2,-1,1/4|0,1/4,-1>^+; b:=<45/2,20+35/4,105/4>;
-A^(-1).b; evalf(%);
```

The solution can also be obtained by hand from the augmented matrix of A and $-\vec{b}$, using the linear algebra toolkit of swap, combination and multiply.

Problem 2(c). The homogeneous problem is $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$. It can be solved by a variety of methods, three major methods enumerated below. Choose a method and solve for $\vec{x}(t)$.

Answer 2(c): The homogenous scalar general solution is

$$\begin{aligned}x_1(t) &= -\frac{1}{2}c_1e^{-t} + 2c_2e^{-at} + 2c_3e^{-bt}, \\x_2(t) &= -\sqrt{5}c_2e^{-at} + \sqrt{5}c_3e^{-bt}, \\x_3(t) &= c_1e^{-t} + c_2e^{-at} + c_3e^{-bt}.\end{aligned}$$

Three Methods for Solving $\vec{u}' = A\vec{u}$

- **Eigenanalysis Method.** Three eigenpairs of matrix A are required. The matrix A must be diagonalizable, meaning there are 3 eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), (\lambda_3, \vec{v}_3)$. The main theorem says that the general solution of $\vec{u}' = A\vec{u}$ is

$$\vec{u}(t) = c_1e^{\lambda_1 t}\vec{v}_1 + c_2e^{\lambda_2 t}\vec{v}_2 + c_3e^{\lambda_3 t}\vec{v}_3.$$

- **Laplace's Method.** Solve the scalar equations by the Laplace transform method. The **resolvent method** automates this process: $\vec{u}(t) = \mathcal{L}^{-1}((sI - A)^{-1})\vec{u}(0)$.
- **Cayley-Hamilton-Ziebur Method.** The solution $\vec{u}(t)$ is a vector linear combination of the Euler solution atoms found from the roots of the characteristic equation $|A - \lambda I| = 0$. The vectors in the linear combination are determined by an explicit formula.

Solution Details for Problem 2(c)

The roots of the characteristic polynomial are used in all three methods. This is the polynomial equation $|A - \lambda I| = 0$, having n roots real and complex, for an $n \times n$ matrix A .

Subtract λ from the diagonal of A and form the determinant. Then cofactor expansion on row 3 gives

$$|A - \lambda I| = \begin{vmatrix} -1 - \lambda & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 - \lambda & \frac{1}{4} \\ 0 & \frac{1}{4} & -1 - \lambda \end{vmatrix} = (-1 - \lambda) \left(-\frac{1}{16} + (-1 - \lambda)^2 - \frac{1}{4} \right).$$

The roots are $-1, -a, -b$ where $a = 1 + \sqrt{5}/4 = 1.56, b = 1 - \sqrt{5}/4 = 0.44$. The three roots are distinct, real and negative.

Eigenanalysis Method

The eigenpairs must be found, in order to assemble the solution vector $\vec{u}(t)$. Technology can be used to find the answers, which are

$$\left(-1, \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix} \right), \quad \left(-a, \begin{pmatrix} 2 \\ -\sqrt{5} \\ 1 \end{pmatrix} \right), \quad \left(-b, \begin{pmatrix} 2 \\ \sqrt{5} \\ 1 \end{pmatrix} \right).$$

Without technology, there are three homogeneous problems to solve of the form $B\vec{v} = \vec{0}$, for eigenvector \vec{v} . Enumerated, they are:

Case $\lambda = -1$. Then $B = A + I = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$

Case $\lambda = -a$. Then $B = A + aI = \begin{pmatrix} \frac{\sqrt{5}}{4} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{5}}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{\sqrt{5}}{4} \end{pmatrix}$

Case $\lambda = -b$. Then $B = A + bI = \begin{pmatrix} -\frac{\sqrt{5}}{4} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{\sqrt{5}}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & -\frac{\sqrt{5}}{4} \end{pmatrix}$

In each case, the system $B\vec{v} = \vec{0}$ is solved using the last frame algorithm (there is in each case one free variable). The eigenvector reported is the partial derivative of the general solution on the invented symbol t_1 , which was assigned to the free variable.

Application of the Theorem

System $\vec{u}' = A\vec{u}$ is solved in the diagonalizable case in terms of the eigenpairs of A , denoted as $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), (\lambda_3, \vec{v}_3)$. The solution of $\vec{u}' = A\vec{u}$ is given by the formula

$$\vec{u}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + c_3 e^{\lambda_3 t} \vec{v}_3.$$

In the present case, the solution is

$$\vec{u}(t) = c_1 e^{-t} \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix} + c_2 e^{-at} \begin{pmatrix} 2 \\ -\sqrt{5} \\ 1 \end{pmatrix} + c_3 e^{-bt} \begin{pmatrix} 2 \\ \sqrt{5} \\ 1 \end{pmatrix}.$$

Symbols c_1, c_2, c_3 in the solution are arbitrary constants, uniquely determined by initial conditions. In scalar form, the solution is

$$\begin{aligned}x_1(t) &= -\frac{1}{2}c_1e^{-t} + 2c_2e^{-at} + 2c_3e^{-bt}, \\x_2(t) &= -\sqrt{5}c_2e^{-at} + \sqrt{5}c_3e^{-bt}, \\x_3(t) &= c_1e^{-t} + c_2e^{-at} + c_3e^{-bt}.\end{aligned}$$

Laplace Transform Method

The **Laplace Method** for solving $\vec{u}'(t) = A\vec{u}(t)$ is based upon transforming all differential equations into the frequency domain. Then time variable t no longer appears, being replaced by the frequency variable s .

The homogeneous system of differential equations is

$$\begin{aligned}x' &= -x + \frac{y}{2}, \\y' &= \frac{x}{2} - y + \frac{z}{4} + 20 \\z' &= \frac{y}{4} - z.\end{aligned}$$

Transforming to the s -domain uses the parts rule $\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$. Then

$$\begin{aligned}s\mathcal{L}(x) - x(0) &= -\mathcal{L}(x) + \frac{1}{2}\mathcal{L}(y), \\s\mathcal{L}(y) - y(0) &= \frac{1}{2}\mathcal{L}(x) - \mathcal{L}(y) + \frac{1}{4}\mathcal{L}(z) + 20 \\s\mathcal{L}(z) - z(0) &= \frac{1}{4}\mathcal{L}(y) - \mathcal{L}(z).\end{aligned}$$

View these equations as linear algebraic equations for the symbols $\mathcal{L}(x), \mathcal{L}(y), \mathcal{L}(z)$. Move terms left and right to re-write the scalar equations as a matrix system

$$\begin{pmatrix} s+1 & -\frac{1}{2} & 0 \\ \frac{1}{2} & s+1 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & s+1 \end{pmatrix} \begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \\ \mathcal{L}(z) \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}.$$

The system is solved by inverting the coefficient matrix C on the left, using the adjugate formula $C^{-1} = \mathbf{adj}(C)/|C|$. Write the answer as

$$C^{-1}(s) = \begin{pmatrix} s+1 & -\frac{1}{2} & 0 \\ \frac{1}{2} & s+1 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & s+1 \end{pmatrix}^{-1} = \frac{1}{\Delta(s)} \begin{pmatrix} s^2 + 2s + \frac{15}{16} & \frac{s+1}{2} & \frac{1}{8} \\ \frac{s+1}{2} & (s+1)^2 & \frac{s+1}{4} \\ \frac{1}{8} & \frac{s+1}{4} & s^2 + 2s + \frac{3}{4} \end{pmatrix}.$$

Symbol $\Delta(s) = (s+1)(s+a)(s+b)$ is the determinant of $C(s)$. Then

$$\begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \\ \mathcal{L}(z) \end{pmatrix} = C^{-1}(s) \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}.$$

Backward table steps require solving nine equations of the form $\mathcal{L}(f(t)) = \frac{p(s)}{\Delta(s)}$. A computer algebra system reduces the effort, able to write $C^{-1}(s) = \mathcal{L}(\Phi(t))$, using symbols $f_1 = e^{-t}, f_2 = e^{-at}, f_3 = e^{-bt}$, where

$$\Phi(t) = \begin{pmatrix} \frac{1}{5}f_1 + \frac{2}{5}f_2 + \frac{2}{5}f_3 & \frac{1}{\sqrt{5}}(f_3 - f_2) & \frac{1}{5}f_2 + \frac{1}{5}f_3 - \frac{2}{5}f_1 \\ \frac{1}{\sqrt{5}}(f_3 - f_2) & \frac{1}{2}f_2 + \frac{1}{2}f_3 & \frac{1}{2\sqrt{5}}(f_3 - f_2) \\ \frac{1}{5}f_2 + \frac{1}{5}f_3 - \frac{2}{5}f_1 & \frac{1}{2\sqrt{5}}(f_3 - f_2) & \frac{4}{5}f_1 + \frac{1}{10}f_2 + \frac{1}{10}f_3 \end{pmatrix}$$

Then $\mathcal{L}(\vec{u}(t)) = \mathcal{L}(\Phi(t)\vec{u}(0))$ implies by Lerch's cancelation law the formula

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{5}f_1 + \frac{2}{5}f_2 + \frac{2}{5}f_3 & \frac{1}{\sqrt{5}}(f_3 - f_2) & \frac{1}{5}f_2 + \frac{1}{5}f_3 - \frac{2}{5}f_1 \\ \frac{1}{\sqrt{5}}(f_3 - f_2) & \frac{1}{2}f_2 + \frac{1}{2}f_3 & \frac{1}{2\sqrt{5}}(f_3 - f_2) \\ \frac{1}{5}f_2 + \frac{1}{5}f_3 - \frac{2}{5}f_1 & \frac{1}{2\sqrt{5}}(f_3 - f_2) & \frac{4}{5}f_1 + \frac{1}{10}f_2 + \frac{1}{10}f_3 \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}.$$

The Resolvent Method

The scalar system solved above is exactly

$$(sI - A)\mathcal{L}(\vec{u}(t)) = \vec{u}(0), \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{4} \\ 0 & \frac{1}{4} & -1 \end{pmatrix}, \quad \vec{u}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}.$$

The system $(sI - A)\mathcal{L}(\vec{u}(t)) = \vec{u}(0)$ is called the **resolvent equation**. The inverse of the coefficient matrix, $(sI - A)^{-1}$, is called the **resolvent matrix**, because $\mathcal{L}(\vec{u}(t)) = (sI - A)^{-1}\vec{u}(0)$. If these statements make sense to you, then please use them to solve problems. Otherwise, ignore the information, and solve problems in the same manner as outlined earlier.

Engineering and Laplace Transforms

Both mechanical engineering and electrical engineering have rich support for Laplace theory. Using Laplace theory has the advantage that many persons can help you through difficult times. Independent persons prefer to choose the method from their own private toolbox.

Cayley-Hamilton-Ziebur Method

The Ziebur Lemma implies that the solution of the system $\vec{u}'(t) = A\vec{u}(t)$ is given by the formula

$$\vec{u}(t) = \vec{d}_1 e^{-t} + \vec{d}_2 e^{-at} + \vec{d}_3 e^{-bt}.$$

THEOREM. Vectors $\vec{d}_1, \vec{d}_2, \vec{d}_3$ are uniquely determined by initial condition $\vec{u}(0)$, which is a column vector of prescribed constants, by the matrix equation.

$$\langle \vec{d}_1 | \vec{d}_2 | \vec{d}_3 \rangle = \langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle (W(0)^T)^{-1}$$

Symbol $W(t)$ is the Wronskian matrix of the three Euler solution atoms. Notation $\langle \vec{A} | \vec{B} | \vec{C} \rangle$ is the augmented matrix of the three columns vectors $\vec{A}, \vec{B}, \vec{C}$.

Illustration. For the heating example, with $a = 1 + \sqrt{5}/4 = 1.56, b = 1 - \sqrt{5}/4 = 0.44$, the Euler solution atoms are e^{-t}, e^{-at}, e^{-bt} . The Wronskian matrix is

$$W(t) = \begin{pmatrix} e^{-t} & e^{-at} & e^{-bt} \\ -e^{-t} & -ae^{-at} & -be^{-bt} \\ e^{-t} & a^2 e^{-at} & b^2 e^{-bt} \end{pmatrix}, \quad W(0) = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -a & -b \\ 1 & a^2 & b^2 \end{pmatrix}.$$

$$\text{Then } (W(0)^T)^{-1} = \begin{pmatrix} -\frac{11}{5} & \frac{8}{5} - \frac{2}{5}\sqrt{5} & \frac{8}{5} + \frac{2}{5}\sqrt{5} \\ -\frac{32}{5} & \frac{16}{5} - \frac{2}{5}\sqrt{5} & \frac{16}{5} + \frac{2}{5}\sqrt{5} \\ -\frac{16}{5} & \frac{8}{5} & \frac{8}{5} \end{pmatrix}.$$

For initial state $\vec{u}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle = \begin{pmatrix} 1 & -1 & 5/4 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1/8 \end{pmatrix}$. Then

$$\vec{u}(t) = \langle \vec{d}_1 | \vec{d}_2 | \vec{d}_3 \rangle \begin{pmatrix} e^{-t} \\ e^{-at} \\ e^{-bt} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 5/4 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1/8 \end{pmatrix} \begin{pmatrix} -\frac{11}{5} & \frac{8}{5} - \frac{2}{5}\sqrt{5} & \frac{8}{5} + \frac{2}{5}\sqrt{5} \\ -\frac{32}{5} & \frac{16}{5} - \frac{2}{5}\sqrt{5} & \frac{16}{5} + \frac{2}{5}\sqrt{5} \\ -\frac{16}{5} & \frac{8}{5} & \frac{8}{5} \end{pmatrix} \begin{pmatrix} e^{-t} \\ e^{-at} \\ e^{-bt} \end{pmatrix}$$

For general initial state $\vec{u}(0) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$,

$$\langle \vec{u}(0) | A \vec{u}(0) | A^2 \vec{u}(0) \rangle = \begin{pmatrix} c_1 & -c_1 + \frac{1}{2}c_2 & \frac{5}{4}c_1 - c_2 + \frac{1}{8}c_3 \\ c_2 & \frac{1}{2}c_1 - c_2 + \frac{1}{4}c_3 & -c_1 + \frac{21}{16}c_2 - \frac{1}{2}c_3 \\ c_3 & \frac{1}{4}c_2 - c_3 & \frac{1}{8}c_1 - \frac{1}{2}c_2 + \frac{17}{16}c_3 \end{pmatrix}.$$

Then $\vec{u}(t)$ is this matrix times $(W(0)^T)^{-1}$ times the column vector of atoms e^{-t}, e^{-at}, e^{-bt} .

Details for the Theorem

The idea for finding the three vectors is differentiation of Ziebur's equation, two times, to get three equations

$$\begin{aligned} \vec{u}(t) &= \vec{d}_1 e^{-t} + \vec{d}_2 e^{-at} + \vec{d}_3 e^{-bt}, \\ \vec{u}'(t) &= -\vec{d}_1 e^{-t} - a\vec{d}_2 e^{-at} - b\vec{d}_3 e^{-bt}, \\ \vec{u}''(t) &= \vec{d}_1 e^{-t} + a^2\vec{d}_2 e^{-at} + b^2\vec{d}_3 e^{-bt}. \end{aligned}$$

Identities $\vec{u}'(t) = A\vec{u}(t)$ and $\vec{u}''(t) = A\vec{u}'(t) = AA\vec{u}(t)$ imply that the left sides are simplified to

$$\begin{aligned} \vec{u}(t) &= \vec{d}_1 e^{-t} + \vec{d}_2 e^{-at} + \vec{d}_3 e^{-bt}, \\ A\vec{u}(t) &= -\vec{d}_1 e^{-t} - a\vec{d}_2 e^{-at} - b\vec{d}_3 e^{-bt}, \\ A^2\vec{u}(t) &= \vec{d}_1 e^{-t} + a^2\vec{d}_2 e^{-at} + b^2\vec{d}_3 e^{-bt}. \end{aligned}$$

The critical idea is to substitute $t = 0$, which because of $e^0 = 1$ gives the following three equations for unknowns $\vec{d}_1, \vec{d}_2, \vec{d}_3$:

$$\begin{aligned} \vec{u}(0) &= \vec{d}_1 + \vec{d}_2 + \vec{d}_3, \\ A\vec{u}(0) &= -\vec{d}_1 - a\vec{d}_2 - b\vec{d}_3, \\ A^2\vec{u}(0) &= \vec{d}_1 + a^2\vec{d}_2 + b^2\vec{d}_3. \end{aligned}$$

How to solve these vector equations for unknowns $\vec{d}_1, \vec{d}_2, \vec{d}_3$? To begin, solve the scalar system

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -a & -b \\ 1 & a^2 & b^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

where variables x, y, z are the first components of $\vec{d}_1, \vec{d}_2, \vec{d}_3$, and similarly, b_1, b_2, b_3 are the first components of vector $\vec{u}(0), A\vec{u}(0), A^2\vec{u}(0)$:

$$\begin{aligned} x &= \vec{d}_1 \cdot \vec{v}, & b_1 &= \vec{u}(0) \cdot \vec{v}, \\ y &= \vec{d}_2 \cdot \vec{v}, & b_2 &= A\vec{u}(0) \cdot \vec{v}, \\ z &= \vec{d}_3 \cdot \vec{v}, & b_3 &= A^2\vec{u}(0) \cdot \vec{v}, \end{aligned} \quad \vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The equations also apply to find the second components, using $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then the third com-

ponents using $\vec{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The three systems of equations can be written as one huge matrix equation:

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -a & -b \\ 1 & a^2 & b^2 \end{pmatrix} \langle \vec{d}_1 | \vec{d}_2 | \vec{d}_3 \rangle^T = \langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle^T$$

Taking the transpose across the equation gives

$$\langle \vec{d}_1 | \vec{d}_2 | \vec{d}_3 \rangle \begin{pmatrix} 1 & 1 & 1 \\ -1 & -a & -b \\ 1 & a^2 & b^2 \end{pmatrix}^T = \langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle$$

Finally, invert the matrix $W(0)^T$ and multiply across the equation on the right to obtain

$$\langle \vec{d}_1 | \vec{d}_2 | \vec{d}_3 \rangle = \langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle \left(\begin{pmatrix} 1 & 1 & 1 \\ -1 & -a & -b \\ 1 & a^2 & b^2 \end{pmatrix}^T \right)^{-1}$$

This is exactly the equation reported in the theorem,

$$\langle \vec{d}_1 | \vec{d}_2 | \vec{d}_3 \rangle = \langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle \left(W(0)^T \right)^{-1}$$

It has been observed that if $f_1 = e^{-t}$, $f_2 = e^{-at}$, $f_3 = e^{-bt}$ are replaced by a new basis of solutions such that $W(0) = I$, then $\vec{d}_1 = \vec{u}(0)$, $\vec{d}_2 = A\vec{u}(0)$, $\vec{d}_3 = A^2\vec{u}(0)$. The resulting solution in this case is

$$\vec{u}(t) = f_1(t)\vec{u}(0) + f_2(t)A\vec{u}(0) + f_3(t)A^2\vec{u}(0).$$

Which Method is the Best?

The eigenanalysis method seems to be the best, because it is a method for simplifying coordinates, hence a shorter answer. Except in the case of complex roots. Or in the case that the matrix A fails to be diagonalizable. In practice, the method used to solve the equation $\vec{u}'(t) = A\vec{u}(t)$ has to be tuned to the expected application. Dynamical systems are an important example. For dynamical systems, the actual solution is less important than its formula, which is a linear combination of Euler solution atoms, according to Cayley-Hamilton-Ziebur.

Laplace theory provides a simple formula for the solution of $\vec{u}'(t) = A\vec{u}(t)$. It has the form

$$\vec{u}(t) = \Phi(t)\vec{u}(0).$$

The matrix $\Phi(t)$ in the Laplace formula is computed from $\mathcal{L}(\Phi(t)) = (sI - A)^{-1}$. Although this computation is nontrivial by hand, computer algebra system automation is possible.

Matrix $\Phi(t)$ is called the **Exponential Matrix**, denoted by e^{At} . Computer algebra system **Maple** computes $\Phi(t)$ by the command `LinearAlgebra[MatrixExponential](A, t)`. Both **Matlab** and **Mathematica** support symbolic computation of the matrix exponential $\Phi(t)$.