Problem 9. Solving Higher Order Constant-Coefficient Equations

The Algorithm applies to constant-coefficient homogeneous linear differential equations of order N, for example equations like

$$y'' + 16y = 0, \quad y'''' + 4y'' = 0, \quad \frac{d^5y}{dx^5} + 2y''' + y'' = 0.$$

1. Find the Nth degree characteristic equation by Euler's substitution $y = e^{rx}$. For instance, $y'' + 16y = 0$ has characteristic equation $r^2 + 16 = 0$, a polynomial equation of degree $N = 2$.
2. Find all real roots and all complex conjugate pairs of roots satisfying the characteristic equation. List the N roots according to multiplicity.
3. Construct N distinct Euler solution atoms from the list of roots. Then the general solution of the differential equation is a linear combination of the Euler solution atoms with arbitrary coefficients $c_1, c_2, c_3, \ldots$.
   The solution space is then $S = \textbf{span}$(the N Euler solution atoms).

Examples: Constructing Euler Solution Atoms from roots.

Three roots $0, 0, 0$ produce three atoms $e^{0x}, xe^{0x}, x^2e^{0x}$ or $1, x, x^2$.

Three roots $0, 0, 2$ produce three atoms e^{0x}, xe^{0x}, e^{2x}.

Two complex conjugate roots $2 \pm 3i$ produce two atoms $e^{2x}\cos(3x), e^{2x}\sin(3x)$.

> **Explained**. The Euler substitution $y = e^{rx}$ produces a solution of the differential equation when r is a complex root of the characteristic equation. Complex exponentials are not used directly. Ever. They are replaced by sines and cosines times real exponentials, which are Euler solution atoms. Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ implies $e^{2x}\cos(3x) = e^{2x}\frac{e^{3xi}+e^{-3xi}}{2} = \frac{1}{2}e^{2x+3xi} + \frac{1}{2}e^{2x-3xi}$, which is a linear combination of complex exponentials, solutions of the differential equation because of Euler's substitution. Superposition implies $e^{2x}\cos(3x)$ is a solution. Similar for $e^{2x}\sin(3x)$. The independent pair $e^{2x}\cos(3x), e^{2x}\sin(3x)$ replaces both $e^{(2+3i)x}$ and $e^{(2-3i)x}$.

Four complex conjugate roots listed according to multiplicity as $2\pm 3i, 2\pm 3i$ produce four atoms $e^{2x}\cos(3x), e^{2x}\sin(3x), xe^{2x}\cos(3x), xe^{2x}\sin(3x)$.

Seven roots $1, 1, 3, 3, 3, \pm 3i$ produce seven atoms $e^x, xe^x, e^{3x}, xe^{3x}, x^2e^{3x}, \cos(3x), \sin(3x)$.

Two conjugate complex roots $a \pm bi$ ($b > 0$) arising from roots of $(r-a)^2 + b^2 = 0$ produce two atoms $e^{ax}\cos(bx), e^{ax}\sin(bx)$.

The Problem

Solve for the general solution or the particular solution satisfying initial conditions.

(a) $y'' + 4y' = 0$

(b) $y'' + 4y = 0$

(c) $y''' + 4y' = 0$

(d) $y'' + 4y = 0,\ y(0) = 1,\ y'(0) = 2$

(e) $y'''' + 81y'' = 0,\ y(0) = y'(0) = 0,\ y''(0) = y'''(0) = 1$

(f) The characteristic equation is $(r+1)^2(r^2-1) = 0$.

(g) The characteristic equation is $(r-1)^2(r^2-1)^2((r+1)^2+9) = 0$.

(h) The characteristic equation roots, listed according to multiplicity, are $0, 0, -1, 2, 2, 3+4i, 3-4i, 3+4i, 3-4i$.

Problem 10. Laplace Theory

> Laplace theory collects theorems and transform tables to implement the *method of quadrature* for higher order differential equations, linear systems of differential equations, and certain partial differential equations.
>
> Laplace's method *solves differential equations*.

Laplace's Quadrature Method: multiply the equation by the Laplace integrator $e^{-st}\,dt$ and then integrate across the equation $t = 0$ to $t = \infty$.

Laplace's Method: multiply across the equation by symbol $\mathcal{L}$, then manipulate the result as though $\mathcal{L}$ is a matrix. Solve for the unknown(s) using Laplace tables and Laplace theorems.

Laplace Theory uses properties and tables, and almost never the **Direct Laplace Transform** $F(s) = \int_0^\infty f(t)e^{-st}\,dt$. See Sample Problem 10 for techniques and ideas which apply to the problems below.

The Problem. Solve by table methods or Laplace's method.

(a) Forward table. Find $\mathcal{L}(f(t))$ for $f(t) = 3(t+1)^2e^{2t} + 2e^t\sin(3t)$.

(b) Backward table. Find $f(t)$ for

$$\mathcal{L}(f(t)) = \frac{4s}{s^2+4} + \frac{s-1}{s^2-2s+5}.$$

(c) Solve the initial value problem $x''(t) + 2x'(t) + 5x(t) = e^t$, $x(0) = 0$, $x'(0) = 1$.