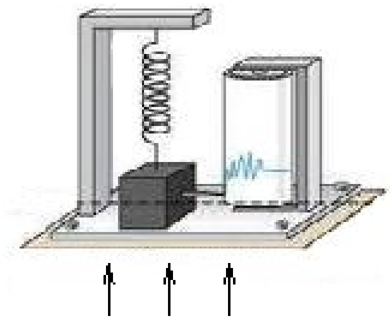
Problem 5. Vertical Motion Seismoscope

The 1875 **horizontal motion seismoscope** of F. Cecchi (1822-1887) reacted to an earthquake. It started a clock, and then it started motion of a recording surface, which ran at a speed of 1 cm per second for 20 seconds. The clock provided the observer with the earthquake hit time.



A Simplistic Vertical Motion Seismoscope

The apparently stationary heavy mass on a spring writes with the attached stylus onto a rotating drum, as the ground moves up. The generated trace is $x(t)$.

The motion of the heavy mass m in the figure can be modeled initially by a forced spring-mass system with damping. The initial model has the form

$$mx'' + cx' + kx = f(t)$$

where $f(t)$ is the vertical ground force due to the earthquake. In terms of the vertical ground motion $u(t)$, we write via Newton's second law the force equation $f(t) = -mu''(t)$ (compare to falling body $-mg$). The final model for the motion of the mass is then

$$(1) \quad \begin{cases} x''(t) + 2\beta\Omega_0 x'(t) + \Omega_0^2 x(t) = -u''(t), \\ \dfrac{c}{m} = 2\beta\Omega_0, \quad \dfrac{k}{m} = \Omega_0^2, \\ x(t) = \text{center of mass position measured from equilibrium,} \\ u(t) = \text{vertical ground motion due to the earthquake.} \end{cases}$$

Terms **seismoscope, seismograph, seismometer** refer to the device in the figure. Some observations:

Slow ground movement means $x' \approx 0$ and $x'' \approx 0$, then (1) implies $\Omega_0^2 x(t) = -u''(t)$. The seismometer records ground acceleration.

Fast ground movement means $x \approx 0$ and $x' \approx 0$, then (1) implies $x''(t) = -u''(t)$. The seismometer records ground displacement.

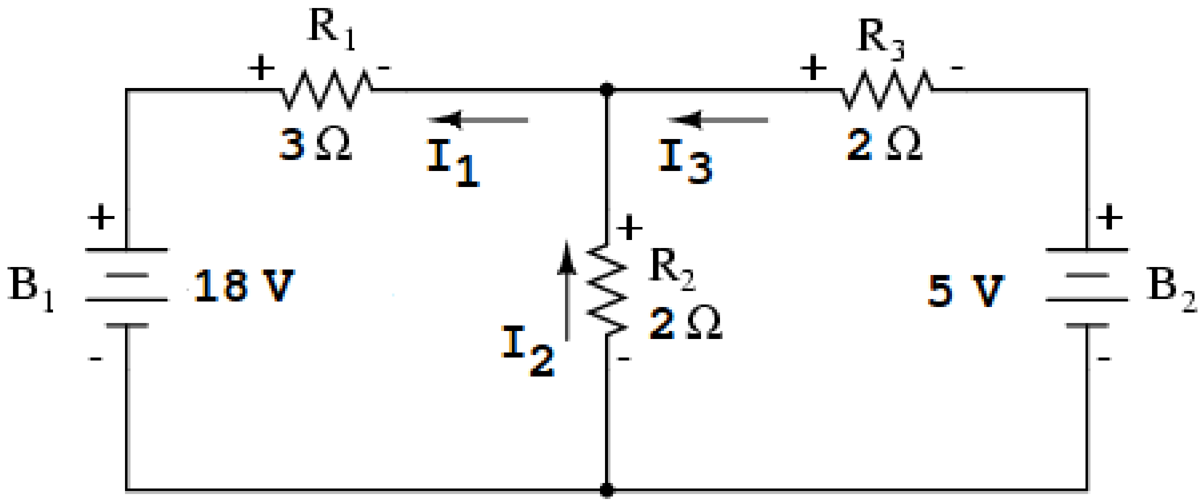
A **release test** begins by starting a vibration with u identically zero. Two successive maxima $(t_1, x_1), (t_2, x_2)$ are recorded. This experiment determines constants β, Ω_0.

The objective of (1) is to determine $u(t)$, by knowing $x(t)$ from the seismograph.

The Problem.

Assume the seismograph trace can be modeled at time $t = 0$ (a time after the earthquake struck) by $x(t) = 10\cos(3t)$. Assume a release test determined $2\beta\Omega_0 = 16$ and $\Omega_0^2 = 80$. Explain how to find a formula for the ground motion $u(t)$, then provide details for the answer $u(t) = \frac{710}{9}\cos(3t) - \frac{160}{3}\sin(3t)$ (assume integration constants are zero).

Problem 6. **Resistive Network** with 2 Loops and DC Sources.



The **Branch Current Method** can be used to find a 3×3 linear system for the **branch currents** I_1, I_2, I_3.

$$\begin{array}{rcrcrcrl} I_1 & - & I_2 & - & I_3 & = & 0 & \text{KCL, upper node} \\ 3I_1 & + & 2I_2 & & & = & 18 & \text{KVL, left loop} \\ & & 2I_2 & - & 2I_3 & = & 5 & \text{KVL, right loop} \end{array}$$

Symbol **KCL** means *Kirchhoff's Current Law*, which says the algebraic sum of the currents at a node is zero. Symbol **KVL** means *Kirchhoff's Voltage Law*, which says the algebraic sum of the voltage drops around a closed loop is zero.

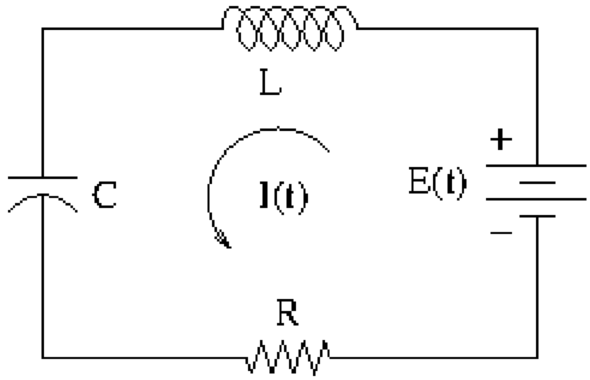
(a) Solve the equations to find the currents I_1, I_2, I_3 in the figure.

(b) Compute the voltage drops across resistors R_1, R_2, R_3. Answer: $\frac{93}{8}, \frac{51}{8}, \frac{11}{8}$ volts.

(c) Replace the 5 volt battery by a 4 volt battery. Solve the system again, and report the new currents and voltage drops.

References. Edwards-Penney 3.7, electric circuits. All About Circuits Volume I – DC, by T. Kuphaldt:
http://www.allaboutcircuits.com/.
Course slides on Electric Circuits:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/electricalCircuits.pdf`
Solved examples of electrical networks can be found in the lecture notes of Ruye Wang:
http://fourier.eng.hmc.edu/e84/lectures/ch2/node2.html.

Problem 7. *RLC*-**Circuit**



The Problem. Suppose $E = \sin(40t)$, $L = 1$ H, $R = 50\,\Omega$ and $C = 0.01$ F. The model for the charge $Q(t)$ is $LQ'' + RQ' + \frac{1}{C}Q = E(t)$.

(a) Differentiate the charge model and substitute $I = \frac{dQ}{dt}$ to obtain the current model $I'' + 50I' + 100I = 40\cos(40t)$.

(b) Find the **reactance** $S = \omega L - \frac{1}{\omega C}$, where $\omega = 40$ is the input frequency, the natural frequency of $E = \sin(40t)$ and $E' = 40\cos(40t)$. Then find the **impedance** $Z = \sqrt{S^2 + R^2}$.

(c) The steady-state current is $I(t) = A\cos(40t) + B\sin(40t)$ for some constants A, B. Substitute $I = A\cos(40t) + B\sin(40t)$ into the current model (a) and solve for A, B.

Answers: $A = -\frac{6}{625}$, $B = \frac{8}{625}$.

(d) Write the answer in (c) in phase-amplitude form $I = I_0 \sin(40t - \delta)$ with $I_0 > 0$ and $\delta \geq 0$. Then compute the **time lag** δ/ω.

Answers: $I_0 = 0.016$, $\delta = \arctan(0.75)$, $\delta/\omega = 0.0160875$.

References

Course slides on Electric Circuits:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/electricalCircuits.pdf`

Edwards-Penney *Differential Equations and Boundary Value Problems*, sections 3.4, 3.5, 3.6, 3.7.