## Problem 7. Picard–Lindelöf Theorem and Spring-Mass Models

**Picard-Lindelöf Theorem**. Let $\vec{f}(x, \vec{y})$ be defined for $|x - x_0| \le h$, $\|\vec{y} - \vec{y}_0\| \le k$, with $\vec{f}$ and $\frac{\partial \vec{f}}{\partial \vec{y}}$ continuous. Then for some constant $H$, $0 < H < h$, the problem

$$\begin{cases} \vec{y}\,'(x) &= \vec{f}(x, \vec{y}(x)), \quad |x - x_0| < H, \\ \vec{y}(x_0) &= \vec{y}_0 \end{cases}$$

has a unique solution $\vec{y}(x)$ defined on the smaller interval $|x - x_0| < H$.

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**The Problem**. The second order problem

$$\begin{cases} u'' + 2u' + 17u = 100, \\ u(0) = 1, \\ u'(0) = -1 \end{cases} \tag{1}$$

is a spring-mass model with damping and constant external force. The variables are time $x$ in seconds and elongation $u(x)$ in meters, measured from equilibrium. Coefficients in the equation represent mass $m = 1$ kg, a viscous damping constant $c = 2$, Hooke's constant $k = 17$ and external force $F(x) = 100$.

Convert the scalar initial value problem into a vector problem, to which Picard's vector theorem applies, by supplying details for the parts below.

**(a)** The conversion uses the **position-velocity substitution** $y_1 = u(x), y_2 = u'(x)$, where $y_1, y_2$ are the invented components of vector $\vec{y}$. Then the initial data $u(0) = 1, u'(0) = -1$ converts to the vector initial data

$$\vec{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

**(b)** Differentiate the equations $y_1 = u(x), y_2 = u'(x)$ in order to find the scalar system of two differential equations, known as a **dynamical system**:

$$y_1' = y_2, \quad y_2' = -17y_1 - 2y_2 + 100.$$

**(c)** The derivative of vector function $\vec{y}(x)$ is written $\vec{y}\,'(x)$ or $\frac{d\vec{y}}{dx}(x)$. It is obtained by componentwise differentiation: $\vec{y}\,'(x) = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$. The vector differential equation model of scalar system (1) is

$$\begin{cases} \vec{y}\,'(x) &= \begin{pmatrix} 0 & 1 \\ -17 & -2 \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ 100 \end{pmatrix}, \\ \vec{y}(0) &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{cases} \tag{2}$$

**(d)** System (2) fits the hypothesis of Picard's theorem, using symbols

$$\vec{f}(x, \vec{y}) = \begin{pmatrix} 0 & 1 \\ -17 & -2 \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ 100 \end{pmatrix}, \quad \vec{y}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The components of vector function $\vec{f}$ are continuously differentiable in variables $x, y_1, y_2$, therefore $\vec{f}$ and $\frac{\partial \vec{f}}{\partial \vec{y}}$ are continuous.

**References**. Chapter 2, Edwards-Penney.
Course slides on the Picard and Direction Fields:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/Picard+DirectionFields.pdf`
Course slides on the Picard Theorem:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/picardHigherOrderSuperposition.pdf`
Course slides on the Vector Picard Theorem:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/picardVectorTheorem.pdf`

**Problem 8**. The velocity of a crossbow bolt launched upward from the ground was determined from a video and a speed gun to complete the following table.

| Time $t$ in seconds | Velocity $v(t)$ in ft/sec | Location |
|---|---|---|
| 0.000 | 60 | Ground |
| 1.7 | 0 | Maximum |
| 3.5 | -52 | Near Ground Impact |

**(a)** The bolt velocity can be approximated by a quadratic polynomial

$$z(t) = at^2 + bt + c$$

which reproduces the table data. Find three equations for the coefficients $a, b, c$. Then solve for the coefficients.

**(b)** Assume a linear drag model $v' = -32 - \rho v$. Substitute the polynomial answer $v = z(t)$ of (a) into this differential equation, then substitute $t = 0$ and solve for $\rho \approx 0.11$.

**(c)** Solve the model $w' = -32 - \rho w$, $w(0) = 60$ with $\rho = 0.11$.

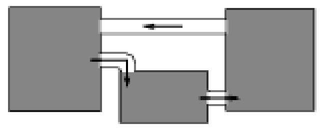
**(d)** The error between $z(t)$ and $w(t)$ can be measured. Is the drag coefficient value $\rho = 0.11$ reasonable?

**References**. Edwards-Penney sections 2.3, 3.1, 3.2.
Course documents on **Linear algebraic equations**:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/linearequDRAFT.pdf`
Course documents on **Newton kinematics**:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/newtonModelsDE2008.pdf`

**Problem 9**. Consider the system of differential equations

$$\begin{aligned} x_1' &= -\tfrac{1}{5}x_1 &&+ \tfrac{1}{7}x_3, \\ x_2' &= \tfrac{1}{5}x_1 - \tfrac{1}{3}x_2, && \\ x_3' &= \tfrac{1}{3}x_2 &&- \tfrac{1}{7}x_3, \end{aligned}$$

for the amounts $x_1, x_2, x_3$ of salt in recirculating brine tanks, as in the figure:



**Recirculating Brine Tanks A, B, C**

The volumes are $50, 30, 70$ for $A, B, C$, respectively.

The steady-state salt amounts in the three tanks are found by formally setting $x_1' = x_2' = x_3' = 0$ and then solving for the symbols $x_1, x_2, x_3$.

**(a)** Solve the corresponding linear system of algebraic equations for answers $x_1, x_2, x_3$.

**(b)** The total amount of salt is uniformly distributed in the tanks in ratio $5:3:7$. Explain this mathematically from the answer in (a).

**References**. Edwards-Penney sections 3.1, 3.2, 7.3 Figure 5.
Course documents on **Linear algebraic equations**:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/linearequDRAFT.pdf`
Course documents on **Systems and Brine Tanks**:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/systemsBrineTank.pdf`