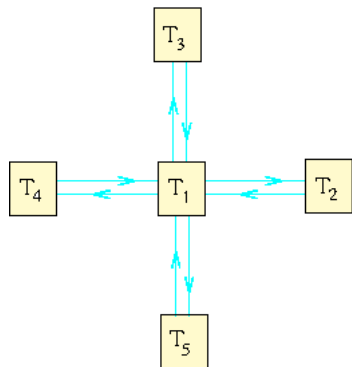# Problem 4. Brine Tanks.



Flow through each pipe is $f$ gallons per unit time.

Each tank has constant volume $V$.

Symbols $x_1(t)$ to $x_5(t)$ are the salt amounts in tanks $T_1$ to $T_5$, respectively.

The differential equations are obtained by the classical balance law, which says that the rate of change in salt amount is the rate in minus the rate out. Individual rates in/out are of the form (flow rate)(salt concentration), where flow rate $f$ has units volume per unit time and $x_i(t)/V$ is the concentration = amount/volume.

$$
\begin{aligned}
x_1'(t) &= \frac{f}{V}\left(x_2(t) + x_3(t) + x_4(t) + x_5(t) - 4x_1(t)\right) \\
x_2'(t) &= \frac{f}{V}\left(x_1(t) - x_2(t)\right), \\
x_3'(t) &= \frac{f}{V}\left(x_1(t) - x_3(t)\right), \\
x_4'(t) &= \frac{f}{V}\left(x_1(t) - x_4(t)\right), \\
x_5'(t) &= \frac{f}{V}\left(x_1(t) - x_5(t)\right).
\end{aligned}
$$

Solve Parts (a) to (e) below.

---

**Part (a).** Change variables $t = Vr/f$ to obtain the new system

$$
\begin{aligned}
\frac{dx_1}{dr} &= x_2 + x_3 + x_4 + x_5 - 4x_1 \\
\frac{dx_2}{dr} &= x_1 - x_2, \\
\frac{dx_3}{dr} &= x_1 - x_3, \\
\frac{dx_4}{dr} &= x_1 - x_4, \\
\frac{dx_5}{dr} &= x_1 - x_5.
\end{aligned}
$$

**Part (b).** Formulate the equations in Part (a) in the system form $\dfrac{d}{dr}\vec{u} = A\vec{u}$.

**Answer:**

$$
A = \begin{pmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}
$$

**Part (c).** Show details for finding the eigenvalues of $A$: $\lambda = 0, -1, -1, -1, -5$.

**Part (d).** Find the eigenvectors of $A$.

**Part (e).** Solve the differential equation $\dfrac{d\vec{u}}{dr} = A\vec{u}$ by the eigenanalysis method.

# Three Methods for Solving $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$

- **Eigenanalysis Method**. The eigenpairs of matrix $A$ are required. The matrix $A$ must be diagonalizable, meaning there are $n$ eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), \ldots, (\lambda_n, \vec{v}_n)$. The main theorem says that the general solution of $\vec{u}' = A\vec{u}$ is

$$\vec{u}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots + c_n e^{\lambda_n t}\vec{v}_n.$$

- **Laplace's Method**. Solve the scalar equations by the Laplace transform method. The **resolvent method** automates this process: $\vec{u}(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)\vec{u}(0)$.

- **Cayley-Hamilton-Ziebur Method**. The solution $\vec{u}(t)$ is a vector linear combination of the Euler solution atoms $f_1, \ldots, f_n$ found from the roots of the characteristic equation $|A - \lambda I| = 0$. The vectors $\vec{d}_1, \ldots, \vec{d}_n$ in the linear combination

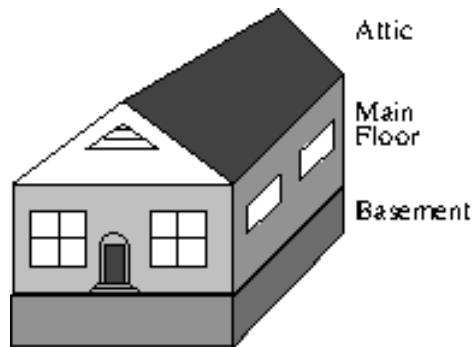$$\vec{u}(t) = f_1(t)\vec{d}_1 + f_2(t)\vec{u}_2 + \cdots + f_n(t)\vec{d}_n$$

are determined by the explicit formula

$$< \vec{d}_1 \,|\, \vec{d}_2 \,|\cdots|\, \vec{d}_n > = < \vec{u}_0 \,|\, A\vec{u}_0 \,|\cdots|\, A^{n-1}\vec{u}_0 > \left(W(0)^T\right)^{-1},$$

where $W(t)$ is the Wronskian matrix of atoms $f_1, \ldots, f_n$ and $\vec{u}_0$ is the initial data.

## Problem 5. Home Heating

Consider a typical home with attic, basement and insulated main floor.



## Heating Assumptions and Variables

- It is usual to surround the main living area with insulation, but the attic area has walls and ceiling without insulation.

- The walls and floor in the basement are insulated by earth.

- The basement ceiling is insulated by air space in the joists, a layer of flooring on the main floor and a layer of drywall in the basement.

The changing temperatures in the three levels is modeled by Newton's cooling law and the variables

$$
\begin{array}{rcl}
z(t) & = & \text{Temperature in the attic,} \\
y(t) & = & \text{Temperature in the main living area,} \\
x(t) & = & \text{Temperature in the basement,} \\
t & = & \text{Time in hours.}
\end{array}
$$

A typical mathematical model is the set of equations

$$
\begin{array}{rcl}
x' & = & \dfrac{3}{4}(45 - x) + \dfrac{1}{4}(y - x), \\[2mm]
y' & = & \dfrac{1}{4}(x - y) + \dfrac{1}{4}(40 - y) + \dfrac{1}{2}(z - y) + 20, \\[2mm]
z' & = & \dfrac{1}{2}(y - z) + \dfrac{1}{2}(35 - z).
\end{array}
$$

Solve Parts (a) to (c) below.

---

**Part (a).** Formulate the system of differential equations as a matrix system $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t) + \vec{b}$. Show details.

**Answer.** $\vec{u} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$, $\quad \vec{b} = \begin{pmatrix} \frac{3}{4}(45) \\ 20 + \frac{40}{4} \\ \frac{35}{2} \end{pmatrix}$, $\quad A = \begin{pmatrix} -1 & \frac{1}{4} & 0 \\ \frac{1}{4} & -1 & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 \end{pmatrix}$

**Part (b).** The heating problem has an **equilibrium solution** $\vec{u}_p(t)$ which is a constant vector of temperatures for the three floors. It is formally found by setting $\frac{d}{dt}\vec{u}(t) = 0$, and then $\vec{u}_p = -A^{-1}\vec{b}$. Justify the algebra and explicitly find $\vec{u}_p(t)$.

**Answer.** $\vec{u}_p(t) = -A^{-1}\vec{b} = \begin{pmatrix} \frac{560}{11} \\ \frac{755}{11} \\ \frac{570}{11} \end{pmatrix} = \begin{pmatrix} 50.91 \\ 68.64 \\ 51.82 \end{pmatrix}$.

**Part (c).** The homogeneous problem is $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$. It can be solved by a variety of methods, three major methods enumerated below. Choose a method and solve for $\vec{x}(t)$.

**Answer**: The homogenous scalar general solution is

$$
\begin{aligned}
x_1(t) &= -2c_1 e^{-t} + \frac{1}{2}c_2 e^{-at} + \frac{1}{2}c_3 e^{-bt}, \\
x_2(t) &= -\frac{1}{2}\sqrt{5}c_2 e^{-at} - \frac{1}{2}\sqrt{5}c_3 e^{-bt}, \\
x_3(t) &= c_1 e^{-t} + c_2 e^{-at} + c_3 e^{-bt}.
\end{aligned}
$$

## Four Methods for Solving $\vec{u}' = A\vec{u}$

- **Eigenanalysis Method**. Three eigenpairs of matrix $A$ are required. The matrix $A$ must be diagonalizable, meaning there are 3 eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), (\lambda_3, \vec{v}_3)$. The main theorem says that the general solution of $\vec{u}' = A\vec{u}$ is

$$
\vec{u}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + c_3 e^{\lambda_3 t}\vec{v}_3.
$$

- **Laplace's Method**. Solve the scalar equations by the Laplace transform method. The **resolvent method** automates this process: $\vec{u}(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)\vec{u}(0)$.

- **Cayley-Hamilton-Ziebur Method**. The solution $\vec{u}(t)$ is a vector linear combination

$$
\vec{u}(t) = \vec{d}_1 f_1(t) + \vec{d}_2 f_2(t) + \vec{d}_3 f_3(t)
$$

  of the Euler solution atoms $f_1, f_2, f_3$ found from the roots of the characteristic equation $|A - \lambda I| = 0$.

  The vectors $\vec{d}_1, \vec{d}_2, \vec{d}_3$ are determined by the explicit formula

$$
< \vec{d}_1 \,|\, \vec{d}_2 \,|\, \vec{d}_3 > = < \vec{u}_0 \,|\, A\vec{u}_0 \,|\, A^2\vec{u}_0 > \left(W(0)^T\right)^{-1},
$$

  where $W(t)$ is the Wronskian matrix of atoms $f_1, f_2, f_3$ and $\vec{u}_0$ is the initial data.

- **Exponential Matrix Method**. The method uses $e^{At}$, which is a fundamental matrix $\Phi(t)$ for the system $\frac{d}{dt}\vec{u} = A\vec{u}$, which satisfies the extra condition $\Phi(0) = I = n \times n$ identity matrix. Then the solution of the system can be written $\vec{u}(t) = e^{At}\vec{u}(0)$. Putzer's method applies to find $e^{At}$ in any dimension $n$. More practical is a computer algebra system. For instance, `maple` finds the exponential matrix by this sample code:

  `A:=<1,2|3,4>; LinearAlgebra[MatrixExponential](A,t);`