Chapters 1-2, Edwards–Penney

Quiz 1. Please prepare your own report on 8x11 paper, handwritten. Work alone or in groups.

Quiz1 Problem 1. An answer check for the differential equation and initial condition

$$\frac{dy}{dx} = k(73 - y(x)), \quad y(0) = 28 \tag{1}$$

requires substitution of the candidate solution $y(x) = 73 - 45\, e^{-kx}$ into the left side (LHS) and right side (RHS), then compare the expressions for equality for all symbols. The process of testing LHS = RHS applies to both the differential equation and the initial condition, making the answer check have **two** presentation panels. Complete the following:

1. Show the two panels in an answer check for initial value problem (1).

2. Relate (1) to a Newton cooling model for warming a 28 F frozen ice cream bar to room temperature 73 F.

3. Let x be the time in minutes. Find the Newton cooling constant k, given the additional information that the ice cream bar reaches 33 F in 5 minutes.

References. Edwards-Penney sections 1.1, 1.4, 1.5. Newton cooling in Serway and Vuille, *College Physics 9/E*, Brooks-Cole (2011), ISBN-10: 0840062060. Newton cooling differential equation $\frac{du}{dt} = -h(u(t) - u_1)$, Math 2280 slide Three Examples. Math 2280 slide on Answer checks.

Quiz1 Problem 2. A 2-ft high conical water urn drains from an orifice 6 inches above the base. The tank drains according to the Torricelli model

$$|y(x)|^2 \frac{dy}{dx} = -0.021\sqrt{|y(x)|}, \quad y(0) = y_0. \tag{2}$$

Symbol $y(x) \geq 0$ is the tank water height in feet above the orifice at time x seconds, while $y_0 \geq 0$ is the water height at time $x = 0$.

Establish these facts about the physical problem.

1. If $y_0 > 0$, then the solution $y(x)$ is uniquely determined and computable by numerical software. Justify using Picard's existence-uniqueness theorem.

2. Solve equation (2) using separation of variables when y_0 is 19 inches, then numerically find the drain time. Check your answer with technology.

References. Edwards-Penney, Picard's theorem 1 section 1.3 and Torricelli's Law section 1.4. Tank draining `Mathematica` demo at Wolfram Research. Carl Schaschke, *Fluid Mechanics: Worked Examples for Engineers*, The Institution of Chemical Engineers (2005), ISBN-10: 0852954980, Chapter 6. Math 2280 slide on Picard and Peano Theorems. Manuscript on applications of first order equations Example 35:
`http://www.math.utah.edu/~gustafso/s2018/2280/lectureslides/2250SciEngApplications.pdf`,

Quiz 2. The problems are due Wednesday, week 3. Please prepare your own report on 8x11 paper, handwritten. You may work in groups. Help is available by telephone, office visit or email, or visit the Math Center in LCB.

Quiz2 Problem 1. Three applications for the Newton cooling equation $y' = -h(y - y_1)$ are considered, where h, y_1 are constants.

(a) Cooling. An apple initially 23 degrees Celsius is placed in a refrigerator at 2 degrees Celsius. The exponential model is the apple temperature $u(t) = 2 + 21e^{-ht}$. Display the differential equation and the initial condition.

(b) Heating. A beef roast initially 8 degrees Celsius is placed in an oven at 190 degrees Celsius. The exponential model is the roast temperature $u(t) = 190 - 182e^{-ht}$. Display the differential equation and the initial condition.

(c) Fish Length. K. L. von Bertalanffy in 1934 modeled the growth of fish using the equation $\frac{dL}{dt} = h(L_\infty - L(t))$. The fish has mature length L_∞ inches, length $L(t)$ while growing, t is in months and h is the growth rate. Given growth data of $L(0) = 0$, $L(1) = 5$, $L(2) = 7$, find the mature length L_∞, the growth rate h and the months to grow to 95% of mature length.

References. Edwards-Penney section 1.5. Course web notes on Newton's linear drag model and Newton cooling. Wikipedia biography of Ludwig von Bertalanffy. Pisces Conservation Ltd Growth Models, especially Gompertz, logistic and von Bertalanffy. Serway and Vuille, *College Physics 9/E*, Brooks-Cole (2011), ISBN-10: 0840062060. The Coffee Cooling Problem, a Wolfram Demonstration by S.M. Binder.

Quiz2 Problem 2. Logistic growth $F(x) = rx(1 - x/M)$ can be used to describe the annual natural growth of a fish stock. Symbol $x(t)$ is the stock biomass in number of fish at the start of month t. The intrinsic growth rate is symbol r. The environmental carrying capacity in stock biomass terms is symbol M.

1. Assume a pond has carrying capacity $M = 780$ thousand fish. If 92% of the the fish survive to maturity, then $r = 0.92$. Display the no-harvesting model $x'(t) = F(x(t))$, using only symbols x and t.

2. Assume constant harvesting $H \geq 0$ to give the model $x'(t) = F(x(t)) - H$. Use the college algebra quadratic formula to find the equilibrium points in terms of symbols r, M, H. Then verify facts **A, B, C** from your answer.

   A. If $H = \frac{rM}{4}$, then there is one equilibrium point $x = \frac{M}{2}$ (a double real root).
   B. If $H > \frac{rM}{4}$, then there is no equilibrium point.
   C. If $0 < H < \frac{rM}{4}$, then there are two equilibrium points.

3. Replace symbols r, M by 0.92 and 780. Create a short filmstrip of 5 hand-drawn phase diagrams for the equation $x'(t) = F(x(t)) - H$ using the successive harvest values

$$H = 0, \frac{1}{4}\left(\frac{rM}{4}\right), \frac{1}{2}\left(\frac{rM}{4}\right), \frac{1}{1}\left(\frac{rM}{4}\right), \frac{11}{10}\left(\frac{rM}{4}\right).$$

   Each phase diagram shows the equilibria and at least 5 threaded solutions, with labels for funnel, spout and node. The graph window is $t = 0$ to 36 months and $x = 0$ to $2M$.

4. Justify a guess for the **maximum sustainable harvest**, based on your 5 diagrams. This is an approximate value for the largest catch H that can be taken over 36 months.

References. Edwards-Penney sections 2.1, 2.2. Course documents: Logistic Equation, Stability, Fish Farming. Also, a logistic investigation in Malaysia by M.F. Laham 2012.

Quiz3 Problem 1. A graphic called a **phase diagram** displays the behavior of all solutions of $u' = F(u)$. A **phase line diagram** is an abbreviation for a direction field on the vertical axis (u-axis). It consists of equilibrium points and signs of $F(u)$ between equilibria. A phase diagram can be created solely from a phase line diagram, using just three drawing rules:

1. Solutions don't cross.
2. Equilibrium solutions are horizontal lines $u = c$. All other solutions are increasing or decreasing.
3. A solution curve can be moved rigidly left or right to create another solution curve.

Use these tools on the equation $u' = (u-1)(u-2)^2(u+2)$ to make a phase line diagram, and then make a phase diagram with at least 8 threaded solutions. Label the equilibria as stable, unstable, funnel, spout, node.

References. Edwards-Penney section 2.2. Course document on Stability,

Quiz3 Problem 2. An autonomous differential equation $\frac{dy}{dx} = F(x)$ with initial condition $y(0) = y_0$ has a formal solution

$$y(x) = y_0 + \int_0^x F(u)du.$$

The integral may not be solvable by calculus methods. In this case, the integral is evaluated numerically to compute $y(x)$ or to plot a graphic. There are three basic numerical methods that apply, the rectangular rule (RECT), the trapezoidal rule (TRAP)and Simpson's rule (SIMP).

Apply the three methods for $F(x) = \cos(x^2)$ and $y_0 = 0$ using step size $h = 0.2$ from $x = 0$ to $x = 1$. Then fill in the blanks in the following table. Use technology if it saves time. Lastly, compare the four data sets in a plot, using technology.

x – values	0.0	0.2	0.4	0.6	0.8	1.0
y – to 10 digits	0.0	0.1999680024	0.3989772129	0.5922705167	0.7678475376	0.9045242379
y – RECT values	0.0	0.2	0.3998400213	0.5972854780		0.9448839943
y – TRAP values	0.0	0.1999200107	0.3985627497		0.7646744186	0.8989142250
y – SIMP values	0.0	0.1999666703	0.3989746144	0.5922670741	0.7678445414	

References. Edwards-Penney Sections 2.4, 2.5, 2.6, because methods Euler, Modified Euler and RK4 reduce to RECT, TRAP, SIMP methods when $f(x, y)$ is independent of y, i.e., an equation $y' = F(x)$. Course document on numerical solution of $y' = F(x)$ RECT, TRAP, SIMP methods. Wolfram Alpha at `http://www.wolframalpha.com/` can do the RECT rule and graphics with input string

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integrate cos(x^2) using left endpoint method with interval width 0.2 from
x=0 to x=1
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Quiz 4

Quiz 4, Problem 1. Picard–Lindelöf Theorem and Spring-Mass Models

Picard-Lindelöf Theorem. Let $\vec{f}(x,\vec{y})$ be defined for $|x-x_0|\le h$, $\|\vec{y}-\vec{y}_0\|\le k$, with $\vec{f}$ and $\frac{\partial \vec{f}}{\partial \vec{y}}$ continuous. Then for some constant H, $0<H<h$, the problem

$$\begin{cases} \vec{y}\,'(x) &= \vec{f}(x,\vec{y}(x)), \quad |x-x_0|<H, \\ \vec{y}(x_0) &= \vec{y}_0 \end{cases}$$

has a unique solution $\vec{y}(x)$ defined on the smaller interval $|x-x_0|<H$.

Emile Picard

The Problem. The second order problem

$$\begin{cases} u''+2u'+17u=100, \\ u(0)=1, \\ u'(0)=-1 \end{cases} \tag{1}$$

is a spring-mass model with damping and constant external force. The variables are time x in seconds and elongation $u(x)$ in meters, measured from equilibrium. Coefficients in the equation represent mass $m=1$ kg, a viscous damping constant $c=2$, Hooke's constant $k=17$ and external force $F(x)=100$.

Convert the scalar initial value problem into a vector problem, to which Picard's vector theorem applies, by supplying details for the parts below.

(a) The conversion uses the **position-velocity substitution** $y_1=u(x), y_2=u'(x)$, where y_1,y_2 are the invented components of vector $\vec{y}$. Then the initial data $u(0)=1, u'(0)=-1$ converts to the vector initial data

$$\vec{y}(0)=\begin{pmatrix}1\\-1\end{pmatrix}.$$

(b) Differentiate the equations $y_1=u(x), y_2=u'(x)$ in order to find the scalar system of two differential equations, known as a **dynamical system**:

$$y_1'=y_2, \quad y_2'=-17y_1-2y_2+100.$$

(c) The derivative of vector function $\vec{y}(x)$ is written $\vec{y}\,'(x)$ or $\frac{d\vec{y}}{dx}(x)$. It is obtained by componentwise differentiation: $\vec{y}\,'(x)=\begin{pmatrix}y_1'\\y_2'\end{pmatrix}$. The vector differential equation model of scalar system (1) is

$$\begin{cases} \vec{y}\,'(x) &= \begin{pmatrix}0&1\\-17&-2\end{pmatrix}\vec{y}(x)+\begin{pmatrix}0\\100\end{pmatrix}, \\ \vec{y}(0) &= \begin{pmatrix}1\\-1\end{pmatrix}. \end{cases} \tag{2}$$

(d) System (2) fits the hypothesis of Picard's theorem, using symbols

$$\vec{f}(x,\vec{y})=\begin{pmatrix}0&1\\-17&-2\end{pmatrix}\vec{y}(x)+\begin{pmatrix}0\\100\end{pmatrix}, \quad \vec{y}_0=\begin{pmatrix}1\\-1\end{pmatrix}.$$

The components of vector function $\vec{f}$ are continuously differentiable in variables x, y_1, y_2, therefore $\vec{f}$ and $\frac{\partial \vec{f}}{\partial \vec{y}}$ are continuous.

Quiz4 Problem 2. The velocity of a crossbow bolt launched upward from the ground was determined from a video and a speed gun to complete the following table.

Time t in seconds	Velocity $v(t)$ in ft/sec	Location
0.000	60	Ground
1.7	0	Maximum
3.5	-52	Near Ground Impact

(a) The bolt velocity can be approximated by a quadratic polynomial

$$v(t) = at^2 + bt + c$$

which reproduces the table data. Find three equations for the coefficients a, b, c. Then solve for the coefficients.

(b) Assume a linear drag model $v' = -32 - \rho v$. Substitute the polynomial answer of (a) into this differential equation, then substitute $t = 0$ and solve for $\rho \approx 0.11$.

(c) Solve the model $w' = -32 - \rho w$, $w(0) = 60$ with $\rho = 0.11$.

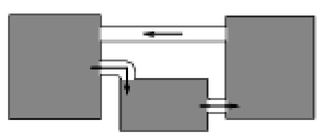
(d) The error between $v(t)$ and $w(t)$ can be measured. Is the drag coefficient value $\rho = 0.11$ reasonable?

References. Edwards-Penney sections 2.3, 3.1, 3.2. Course documents on Linear algebraic equations and Newton kinematics.

Quiz4 Extra Credit Problem 3. Consider the system of differential equations

$$\begin{aligned} x_1' &= -\tfrac{1}{5}x_1 & & + \tfrac{1}{7}x_3, \\ x_2' &= \tfrac{1}{5}x_1 & - \tfrac{1}{3}x_2, & \\ x_3' &= & \tfrac{1}{3}x_2 & - \tfrac{1}{7}x_3, \end{aligned}$$

for the amounts x_1, x_2, x_3 of salt in recirculating brine tanks, as in the figure:



Recirculating Brine Tanks A, B, C

The volumes are $50, 30, 70$ for A, B, C, respectively.

The steady-state salt amounts in the three tanks are found by formally setting $x_1' = x_2' = x_3' = 0$ and then solving for the symbols x_1, x_2, x_3.

(a) Solve the corresponding linear system of algebraic equations for answers x_1, x_2, x_3.

(b) The total amount of salt is uniformly distributed in the tanks in ratio $5 : 3 : 7$. Explain this mathematically from the answer in (a).

References. Edwards-Penney sections 3.1, 3.2, 7.3 Figure 5. Course documents on Linear algebraic equations and Systems and Brine Tanks.

Quiz 5

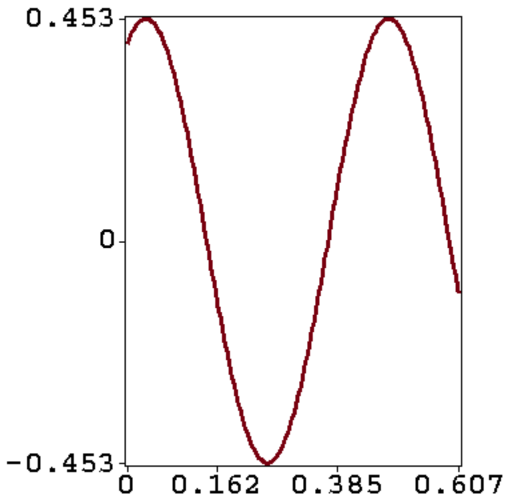
Quiz 5, Problem 1. Harmonic Vibration

A mass of $m = 200$ grams attached to a spring of Hooke's constant k undergoes free undamped vibration. At equilibrium, the spring is stretched 10 cm by a force of 4 Newtons. At time $t = 0$, the spring is stretched 0.4 m and the mass is set in motion with initial velocity 3 m/s directed away from equilibrium. Find:

(a) The numerical value of Hooke's constant k.

(b) The initial value problem for vibration $x(t)$.

(c) Show details for solving the initial value problem for $x(t)$.
The answer is $x(t) = \frac{2}{5}\cos(\sqrt{200}t) + \frac{3}{20}\sqrt{2}\sin(\sqrt{200}t)$, graphed below.



Quiz 5, Problem 2.Harmonic Vibration, Continued

Assume results (a), (b), (c) from Problem 1. In particular, assume

$$x(t) = \frac{2}{5}\cos(\sqrt{200}t) + \frac{3\sqrt{2}}{20}\sin(\sqrt{200}t).$$

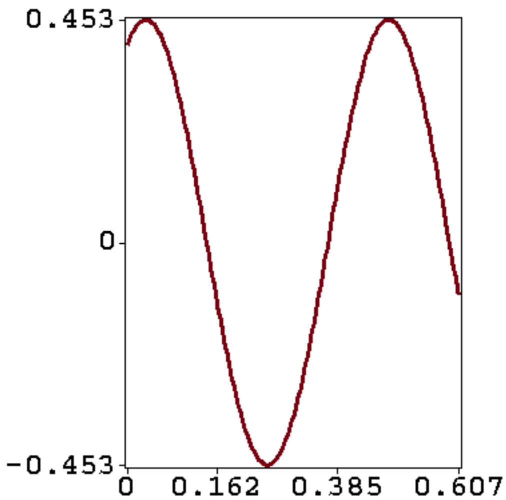
Complete these parts.

(d) Plot the solution $x(t)$ using technology, approximately matching the graphic below.

(e) Show trig details for conversion of $x(t)$ to phase-amplitude form

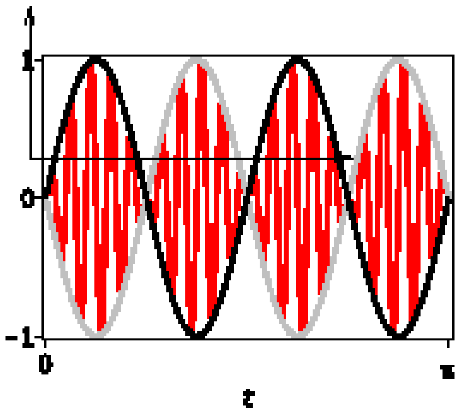
$$x(t) = \frac{\sqrt{82}}{20}\cos(\sqrt{200}t - \arctan(3\sqrt{2}/8)).$$

(f) Report from the answer in part (e) decimal values for the period, amplitude and phase angle. Two-place decimal accuracy is sufficient.



Quiz 5, Problem 3. Beats

The physical phenomenon of **beats** refers to the periodic interference of two sound waves of slightly different frequencies. A destructive interference occurs during a very brief interval, so our impression is that the sound periodically stops, only briefly, and then starts again with a beat, a section of sound that is instantaneously loud again. An illustration of the graphical meaning appears in the figure below.



Beats

Shown in red is a periodic oscillation $x(t) = 2\sin 4t \sin 40t$ with rapidly–varying factor $\sin 40t$ and the two slowly–varying envelope curves $x_1(t) = 2\sin 4t$ (black), $x_2(t) = -2\sin 4t$ (grey).

The undamped, forced spring-mass problem $x'' + 1296x = 640\cos(44t)$, $x(0) = x'(0) = 0$ has by trig identities the solution $x(t) = \cos(36t) - \cos(44t) = 2\sin 4t \sin 40t$.

The Problem. Solve the initial value problem

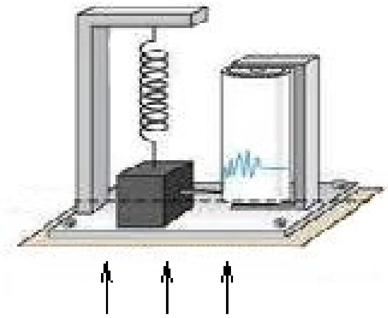
$$x'' + 1444x = 1056\cos(50t), \quad x(0) = x'(0) = 0$$

by undetermined coefficients and linear algebra, obtaining the solution $x(t) = \cos(38t) - \cos(50t)$. Then show the trig details for $x(t) = 2\sin(6t)\sin(44t)$. Finally, graph $x(t)$ and its slowly varying envelope curves on $0 \le t \le \pi$.

Quiz 6

Quiz 6, Problem 1. Vertical Motion Seismoscope

The 1875 **horizontal motion seismoscope** of F. Cecchi (1822-1887) reacted to an earthquake. It started a clock, and then it started motion of a recording surface, which ran at a speed of 1 cm per second for 20 seconds. The clock provided the observer with the earthquake hit time.



A Simplistic Vertical Motion Seismoscope

The apparently stationary heavy mass on a spring writes with the attached stylus onto a rotating drum, as the ground moves up. The generated trace is $x(t)$.

The motion of the heavy mass m in the figure can be modeled initially by a forced spring-mass system with damping. The initial model has the form

$$mx'' + cx' + kx = f(t)$$

where $f(t)$ is the vertical ground force due to the earthquake. In terms of the vertical ground motion $u(t)$, we write via Newton's second law the force equation $f(t) = -mu''(t)$ (compare to falling body $-mg$). The final model for the motion of the mass is then

$$(1) \qquad \begin{cases} x''(t) + 2\beta\Omega_0 x'(t) + \Omega_0^2 x(t) = -u''(t), \\ \dfrac{c}{m} = 2\beta\Omega_0, \quad \dfrac{k}{m} = \Omega_0^2, \\ x(t) = \text{center of mass position measured from equilibrium}, \\ u(t) = \text{vertical ground motion due to the earthquake.} \end{cases}$$

Terms **seismoscope, seismograph, seismometer** refer to the device in the figure. Some observations:

Slow ground movement means $x' \approx 0$ and $x'' \approx 0$, then (1) implies $\Omega_0^2 x(t) = -u''(t)$. The seismometer records ground acceleration.

Fast ground movement means $x \approx 0$ and $x' \approx 0$, then (1) implies $x''(t) = -u''(t)$. The seismometer records ground displacement.

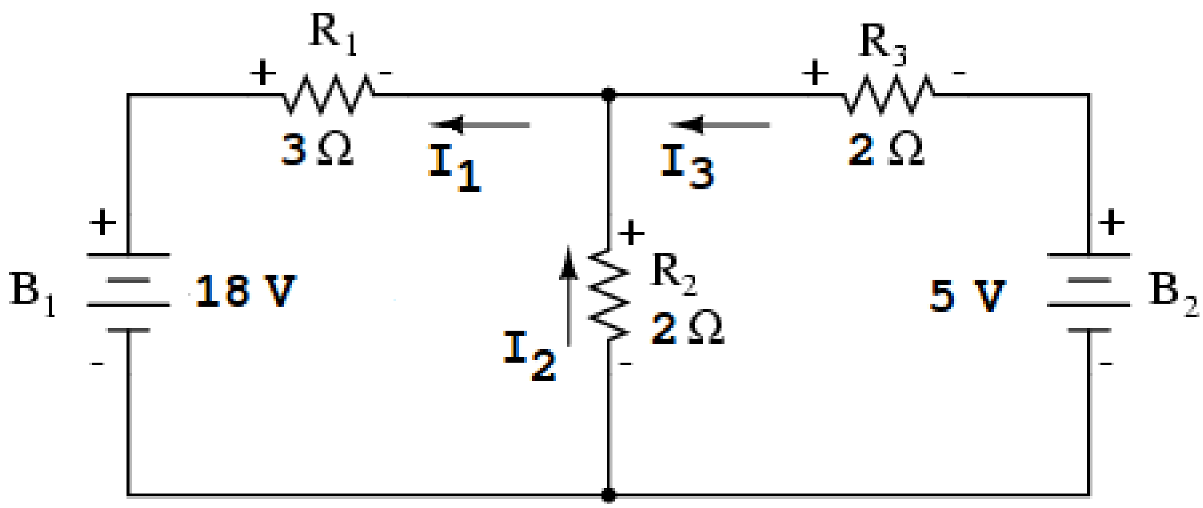
A **release test** begins by starting a vibration with u identically zero. Two successive maxima $(t_1, x_1), (t_2, x_2)$ are recorded. This experiment determines constants β, Ω_0.

The objective of (1) is to determine $u(t)$, by knowing $x(t)$ from the seismograph.

The Problem.

Assume the seismograph trace can be modeled at time $t = 0$ (a time after the earthquake struck) by $x(t) = 10\cos(3t)$. Assume a release test determined $2\beta\Omega_0 = 16$ and $\Omega_0^2 = 80$. Explain how to find a formula for the ground motion $u(t)$, then provide details for the answer $u(t) = \frac{710}{9}\cos(3t) - \frac{160}{3}\sin(3t)$ (assume integration constants are zero).

Quiz6 Problem 2. **Resistive Network** with 2 Loops and DC Sources.



The **Branch Current Method** can be used to find a 3×3 linear system for the **branch currents** I_1, I_2, I_3.

$$\begin{array}{rcrcrcrl} I_1 & - & I_2 & - & I_3 & = & 0 & \text{KCL, upper node} \\ 3I_1 & + & 2I_2 & & & = & 18 & \text{KVL, left loop} \\ & & 2I_2 & - & 2I_3 & = & 5 & \text{KVL, right loop} \end{array}$$

Symbol **KCL** means *Kirchhoff's Current Law*, which says the algebraic sum of the currents at a node is zero. Symbol **KVL** means *Kirchhoff's Voltage Law*, which says the algebraic sum of the voltage drops around a closed loop is zero.

(a) Solve the equations to find the currents I_1, I_2, I_3 in the figure.

(b) Compute the voltage drops across resistors R_1, R_2, R_3. Answer: $\frac{93}{8}, \frac{51}{8}, \frac{11}{8}$ volts.

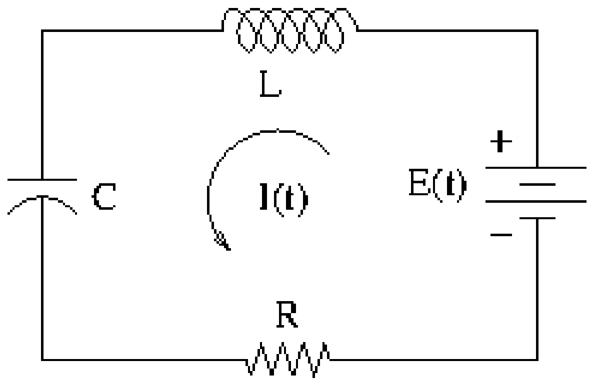
(c) Replace the 5 volt battery by a 4 volt battery. Solve the system again, and report the new currents and voltage drops.

References. Edwards-Penney 3.7, electric circuits. All About Circuits Volume I – DC, by T. Kuphaldt:
http://www.allaboutcircuits.com/.
Course slides on Electric Circuits:
`http://www.math.utah.edu/~gustafso/s2015/2280/electricalCircuits.pdf`.

Solved examples of electrical networks can be found in the lecture notes of Ruye Wang:

http://fourier.eng.hmc.edu/e84/lectures/ch2/node2.html.

Quiz 6, Problem 3. *RLC*-**Circuit**



The Problem. Suppose $E = \sin(40t)$, $L = 1$ H, $R = 50\,\Omega$ and $C = 0.01$ F. The model for the charge $Q(t)$ is $LQ'' + RQ' + \frac{1}{C}Q = E(t)$.

(a) Differentiate the charge model and substitute $I = \frac{dQ}{dt}$ to obtain the current model $I'' + 50I' + 100I = 40\cos(40t)$.

(b) Find the **reactance** $S = \omega L - \frac{1}{\omega C}$, where $\omega = 40$ is the input frequency, the natural frequency of $E = \sin(40t)$ and $E' = 40\cos(40t)$. Then find the **impedance** $Z = \sqrt{S^2 + R^2}$.

(c) The steady-state current is $I(t) = A\cos(40t) + B\sin(40t)$ for some constants A, B. Substitute $I = A\cos(40t) + B\sin(40t)$ into the current model (a) and solve for A, B.

Answers: $A = -\frac{6}{625}$, $B = \frac{8}{625}$.

(d) Write the answer in (c) in phase-amplitude form $I = I_0\sin(40t - \delta)$ with $I_0 > 0$ and $\delta \geq 0$. Then compute the **time lag** δ/ω.

Answers: $I_0 = 0.016$, $\delta = \arctan(0.75)$, $\delta/\omega = 0.0160875$.

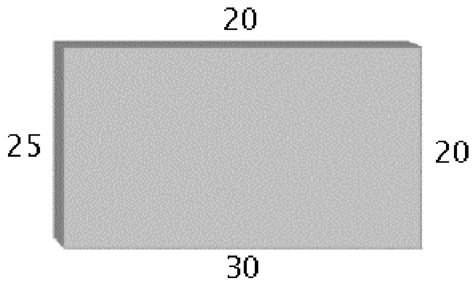
References

Course slides on Electric Circuits:
`http://www.math.utah.edu/~gustafso/s2015/2280/electricalCircuits.pdf`.
Edwards-Penney *Differential Equations and Boundary Value Problems*, sections 3.4, 3.5, 3.6, 3.7.

Extra Credit Quiz7 Problem 1. Heat Transfer and the Mean Value Property.

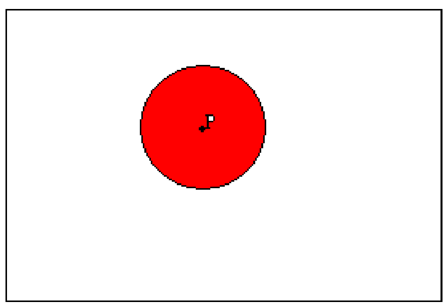
Credit for this problem will cancel missed credit from any quiz score for the entire semester. The score will not affect homework, lab or exam scores.

Consider the cross section of a long rectangular dam on a river, represented in the figure.



The boundaries of the dam are subject to three factors: the temperature in degrees Celsius of the air (20), the water (25), and the ground at its base (30).

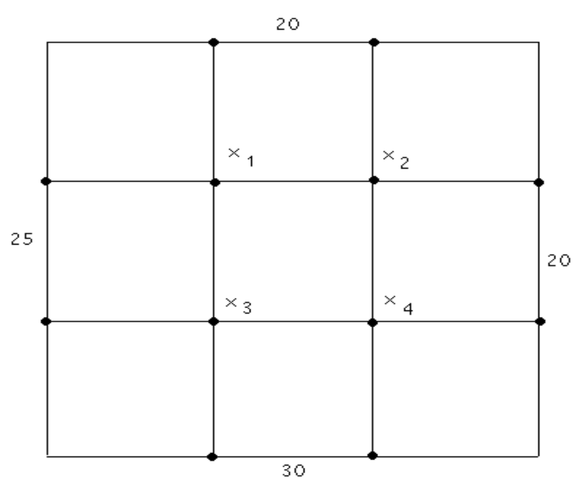
An analysis of the heat transfer from the three sources will be done from the equilibrium temperature, which is found by the Mean Value Property below.



The Mean Value Property

If a plate is at thermal equilibrium, and C is a circle with center P contained in the plate, then the temperature at P is the average value of the temperature function over the boundary of C.

A version of the Mean Value Property says that the temperature at center P of circle C is the average of the temperatures at four equally-spaced points on C. We construct a grid as in the figure below, label the unknown temperatures at interior grid points as x_1, x_2, x_3, x_4, then use the property to obtain four equations.



Four-Point Temperature Averages

$$x_1 = \frac{1}{4}(20 + 25 + x_2 + x_3)$$

$$x_2 = \frac{1}{4}(20 + 20 + x_1 + x_4)$$

$$x_3 = \frac{1}{4}(25 + 30 + x_1 + x_4)$$

$$x_4 = \frac{1}{4}(20 + 30 + x_2 + x_3)$$

The Problem

(a) Solve the equations for the four temperatures $x_1 = 23.125, x_2 = 21.875, x_3 = 25.625, x_4 = 24.375$. Use technology.

(b) Replace the temperatures $20, 25, 30$ by $20, 16, 12$ and re-compute the four temperatures.

(c) Using the temperatures from part (b), subdivide the grid to make it 6×6. Assign 25 unknown temperatures at the interior grid points. Find the equations and solve using technology.

References. EPH Chapters 12, 13, used for Partial Differential Equations 3150.

Extra Credit Quiz7 Problem 2. Archeology and the Dot Product.

Credit for this problem will cancel missed credit from any quiz score for the entire semester. The score will not affect homework, lab or exam scores.

Archeologist Sir Flinders Petrie collected and analyzed pottery fragments from 900 Egyptian graves. He deduced from the data an historical ordering of the 900 sites. Petrie's ideas will be illustrated for 4 sites and 3 pottery types. The matrix rows below represent sites 1, 2, 3, 4 and the matrix columns represent pottery types 1, 2, 3. A matrix entry is 1 if the site has that pottery type and 0 if not. This is an **incidence matrix**.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Petrie Matrix. It is an incidence matrix in which the ones in each column appear together, like the matirx above.

Counting Pottery Types. The dot product of row 2 and row 3 is

$$\begin{pmatrix} 0, 1, 1 \end{pmatrix} \cdot \begin{pmatrix} 1, 0, 1 \end{pmatrix} = 0 * 1 + 1 * 0 + 1 * 1 = 1$$

which means sites 2 and 3 have one pottery type in common. **Please pause on this arithmetic**, until you agree that the products $0 * 1, 1 * 0, 1 * 1$ add to the number of pottery types in common.

Sites with pottery in common are expected to be historically close in time. Because pottery types evolve, old types cease production when newly created pottery types begin production, which gives meaning to the clustered ones in the columns of A.

The Problem. Find a sequence of row swaps which starts with the incidence matrix

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and ends with a Petrie matrix A. Express the swaps as elementary matrices $E_1, E_2, \ldots$ and write A as the product of elementary matrices times C. There is not a unique set of swaps, because of duplicate rows.

You may use Kendall's ideas with Robinson matrices or the simple definition of a Petrie matrix, given above. In any case, after finding Petrie matrix A, compute Robinson's matrix $R = AA^T$ and apply the ideas in sample quiz 7, to double-check the answer.

References. Edwards-Penney Sections 3.4, 3.5, 3.6. David Kendall's 1969 work, Incidence matrices, interval graphs and seriation in archeology. W. S. Robinson, the method for chronologically ordering archeological deposits, in the April 1951 issue of American Antiquity. The incidence matrix C was invented from an archeology exercise on Seriation of 7 Maryland sites based on 3 types of historical ceramics.

Quiz 8

Quiz 8, Problem 1. Solving Higher Order Constant-Coefficient Equations

The Algorithm applies to constant-coefficient homogeneous linear differential equations of order N, for example equations like

$$y'' + 16y = 0, \quad y'''' + 4y'' = 0, \quad \frac{d^5y}{dx^5} + 2y''' + y'' = 0.$$

1. Find the Nth degree characteristic equation by Euler's substitution $y = e^{rx}$. For instance, $y''+16y = 0$ has characteristic equation $r^2+16 = 0$, a polynomial equation of degree $N = 2$.
2. Find all real roots and all complex conjugate pairs of roots satisfying the characteristic equation. List the N roots according to multiplicity.
3. Construct N distinct Euler solution atoms from the list of roots. Then the general solution of the differential equation is a linear combination of the Euler solution atoms with arbitrary coefficients $c_1, c_2, c_3, \ldots$.
   The solution space is then $S = \mathbf{span}(\text{the } N \text{ Euler solution atoms})$.

Examples: Constructing Euler Solution Atoms from roots.

Three roots $0, 0, 0$ produce three atoms $e^{0x}, xe^{0x}, x^2e^{0x}$ or $1, x, x^2$.

Three roots $0, 0, 2$ produce three atoms e^{0x}, xe^{0x}, e^{2x}.

Two complex conjugate roots $2 \pm 3i$ produce two atoms $e^{2x}\cos(3x), e^{2x}\sin(3x)$.

> **Explained**. The Euler substitution $y = e^{rx}$ produces a solution of the differential equation when r is a complex root of the characteristic equation. Complex exponentials are not used directly. Ever. They are replaced by sines and cosines times real exponentials, which are Euler solution atoms. Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ implies $e^{2x}\cos(3x) = e^{2x}\frac{e^{3xi}+e^{-3xi}}{2} = \frac{1}{2}e^{2x+3xi} + \frac{1}{2}e^{2x-3xi}$, which is a linear combination of complex exponentials, solutions of the differential equation because of Euler's substitution. Superposition implies $e^{2x}\cos(3x)$ is a solution. Similar for $e^{2x}\sin(3x)$. The independent pair $e^{2x}\cos(3x), e^{2x}\sin(3x)$ replaces both $e^{(2+3i)x}$ and $e^{(2-3i)x}$.

Four complex conjugate roots listed according to multiplicity as $2\pm 3i, 2\pm 3i$ produce four atoms $e^{2x}\cos(3x), e^{2x}\sin(3x),\ xe^{2x}\cos(3x), xe^{2x}\sin(3x)$.

Seven roots $1, 1, 3, 3, 3, \pm 3i$ produce seven atoms $e^x, xe^x, e^{3x}, xe^{3x}, x^2e^{3x}, \cos(3x), \sin(3x)$.

Two conjugate complex roots $a\pm bi$ ($b > 0$) arising from roots of $(r-a)^2+b^2 = 0$ produce two atoms $e^{ax}\cos(bx), e^{ax}\sin(bx)$.

The Problem

Solve for the general solution or the particular solution satisfying initial conditions.

(a) $y'' + 4y' = 0$

(b) $y'' + 4y = 0$

(c) $y''' + 4y' = 0$

(d) $y'' + 4y = 0,\ y(0) = 1,\ y'(0) = 2$

(e) $y'''' + 81y'' = 0,\ y(0) = y'(0) = 0,\ y''(0) = y'''(0) = 1$

(f) The characteristic equation is $(r + 1)^2(r^2 - 1) = 0$.

(g) The characteristic equation is $(r - 1)^2(r^2 - 1)^2((r + 1)^2 + 9) = 0$.

(h) The characteristic equation roots, listed according to multiplicity, are $0, 0, -1, 2, 2, 3+4i, 3-4i, 3+4i, 3-4i$.

Quiz 8, Problem 2. Laplace Theory

Laplace theory implements the *method of quadrature* for higher order differential equations, linear systems of differential equations, and certain partial differential equations.

Laplace's method **solves differential equations**.

The Problem. Solve by table methods or Laplace's method.

(a) Forward table. Find Ł$(f(t))$ for $f(t) = 3(t+1)^2e^{2t} + 2e^t\sin(3t)$.

(b) Backward table. Find $f(t)$ for

$$Ł(f(t)) = \frac{4s}{s^2+4} + \frac{s-1}{s^2-2s+5}.$$

(c) Solve the initial value problem $x''(t) + 2x'(t) + 5x(t) = e^t$, $x(0) = 0$, $x'(0) = 1$.

Quiz 9

Extra Credit Problem 1. **Piecewise Continuous Inputs**

Consider a passenger SUV on a short trip from Salt Lake City to Evanston, on the Wyoming border. The route is I-80 E, 75 miles through Utah. Google maps estimates 1 hour and 11 minutes driving time. The table below shows the distances, time, road segment and average speed with total trip time 1 hour and 38 minutes. Cities enroute reduce the freeway speed by 10 mph, the trip time effect not shown in the table.

Miles	Minutes	Speed mph	Road Segment	Posted limit mph
18.1	20	54.3	Parley's Walmart to Kimball	65
11.3	12	56.5	Kimball to Wanship	65 – 55
9.1	11	49.6	Wanship to Coalville	70
5.7	6	57	Coalville to Echo Dam	70
16.5	16	61.9	Echo Dam to 75 mph sign	70
39	33	70.9	75 mph sign to Evanston	75

The **velocity function** for the SUV is approximated by

$$V_{\rm pc}(t) = \left\{ \begin{array}{lll} \text{Speed mph} & \text{Time interval minutes} & \text{Road segment} \\ \hline \\ 54.3 & 0 < t < 20 & \text{Parley's Walmart to Kimball} \\ 56.5 & 20 < t < 32 & \text{Kimball to Wanship} \\ 49.6 & 32 < t < 43 & \text{Wanship to Coalville} \\ 57.0 & 43 < t < 49 & \text{Coalville to Echo Dam} \\ 61.9 & 49 < t < 65 & \text{Echo Dam to 75 mph sign} \\ 70.1 & 65 < t < 98 & \text{75 mph sign to Evanston} \end{array} \right.$$

The velocity function $V_{\rm pc}(t)$ is piecewise continuous, because it has the general form

$$f(t) = \begin{cases} f_1(t) & t_1 < t < t_2 \\ f_2(t) & t_2 < t < t_3 \\ \vdots & \vdots \\ f_n(t) & t_n < t < t_{n+1} \end{cases}$$

where functions $f_1, f_2, \ldots, f_n$ are **continuous on the whole real line** $-\infty < t < \infty$. We don't define $f(t)$ at division points, because of many possible ways to make the definition. As long as these values are not used, then it will make no difference. Both right and left hand limits exist at a division point. For Laplace theory, we like the definition $f(0) = \lim_{h \to 0+} f(h)$, which allows the parts rule Ł$(f'(t)) = s$Ł$(f(t)) - f(0)$.

The Problem.

The SUV travels from $t = 0$ to $t = \frac{98}{60} = 1.6$ hours. The odometer trip meter reading $x(t)$ is in miles (assume $x(0) = 0$). The function $V_{\rm pc}(t)$ is an approximation to the speedometer reading. Laplace's method can solve the approximation model

$$\frac{dx}{dt} = V_{\rm pc}(60t), \quad x(0) = 0, \quad x \text{ in miles, } t \text{ in hours,}$$

obtaining $x(t) = \int_0^t V_{\rm pc}(60w)dw$, the same result as the method of quadrature. Show the details. Then display the piecewise linear continuous trip meter reading $x(t)$.

Quiz 11

Background. Switches and Impulses

Laplace's method solves differential equations. It is the premier method for solving equations containing switches or impulses.

Unit Step Define $u(t-a) = \begin{cases} 1 & t \ge a, \\ 0 & t < a. \end{cases}$. It is a **switch**, turned on at $t = a$.

Ramp Define **ramp**$(t-a) = (t-a)u(t-a) = \begin{cases} t-a & t \ge a, \\ 0 & t < a. \end{cases}$, whose graph shape is a continuous **ramp** at 45-degree incline starting at $t = a$.

Unit Pulse Define **pulse**$(t, a, b) = \begin{cases} 1 & a \le t < b, \\ 0 & \text{otherwise} \end{cases} = u(t-a) - u(t-b)$. The switch is **ON** at time $t = a$ and then **OFF** at time $t = b$.

Impulse of a Force

Define the **impulse** of an applied force $F(t)$ on time interval $a \le t \le b$ by the equation

$$\text{Impulse of } F = \int_a^b F(t)dt = \left(\frac{\int_a^b F(t)dt}{b-a}\right)(b-a) = \text{Average Force} \times \text{Duration Time}.$$

Dirac Unit Impulse

A Dirac impulse acts like a hammer hit, a brief injection of energy into a system. It is a special idealization of a real hammer hit, in which only the **impulse** of the force is deemed important, and not its magnitude nor duration.

Define the **Dirac Unit Impulse** by the equation $\delta(t-a) = \dfrac{du}{dt}(t-a)$, where $u(t-a)$ is the unit step. Symbol δ makes sense only under an integral sign, and the integral in question must be a generalized Riemann-Steiltjes integral (definition pending), with new evaluation rules. Symbol δ is an abbreviation like **etc** or **e.g.**, because it abbreviates a paragraph of descriptive text.

- Symbol $M\delta(t-a)$ represents an ideal impulse of magnitude M at time $t = a$. Value M is the change in momentum, but $M\delta(t-a)$ contains no detail about the applied force or the duraction. A common force approximation for a hammer hit of very small duration $2h$ and impulse M is Dirac's approximation
$$F_h(t) = \frac{M}{2h}\,\textbf{pulse}(t, a-h, a+h).$$
- The fundamental equation is $\int_{-\infty}^{\infty} F(x)\delta(x-a)dx = F(a)$. Symbol $\delta(t-a)$ is not manipulated as an ordinary function, but regarded as $du(t-a)/dt$ in a Riemann-Stieltjes integral.

THEOREM (Second Shifting Theorem). Let $f(t)$ and $g(t)$ be piecewise continuous and of exponential order. Then for $a \ge 0$,

Forward table	**Backward table**
$\mathcal{L}(f(t-a)u(t-a)) = e^{-as}\,\mathcal{L}(f(t))$	$e^{-as}\,\mathcal{L}(f(t)) = \mathcal{L}(f(t-a)u(t-a))$
$\mathcal{L}(g(t)u(t-a)) = e^{-as}\,\mathcal{L}\left(g(t)\vert_{t:=t+a}\right)$	$e^{-as}\,\mathcal{L}(f(t)) = \mathcal{L}\left(f(t)u(t)\vert_{t:=t-a}\right)$.

Problem 1. Solve the following by Laplace methods.

(a) Forward table. Compute the Laplace integral for terms involving the unit step, ramp and pulse, in these special cases:

$$(1)\ \mathcal{L}((t-1)u(t-1)) \quad (2)\ \mathcal{L}(e^t\,\textbf{ramp}(t-2)), \quad (3)\ \mathcal{L}(5\,\textbf{pulse}(t,2,4)).$$

(b) Backward table. Find $f(t)$ in the following special cases.

$$(1)\ \mathcal{L}(f) = \frac{e^{-2s}}{s} \quad (2)\ \mathcal{L}(f) = \frac{e^{-s}}{(s+1)^2} \quad (3)\ \mathcal{L}(f) = e^{-s}\frac{3}{s} - e^{-2s}\frac{3}{s}.$$

Problem 2. Solve the following Dirac impulse problem.

(c) Dirac Impulse and the Second Shifting theorem. Solve the following forward table problems.

$$(1)\ \mathcal{L}(2\delta(t-5)), \quad (2)\ \mathcal{L}(2\delta(t-1)+5\delta(t-3)), \quad (3)\ \mathcal{L}(e^t\delta(t-2)).$$

The sum of Dirac impulses in (2) is called an **impulse train**. The numbers 2 and 5 represent the applied **impulse** at times 1 and 3, respectively.

Reference: The Riemann-Stieltjes Integral

Definition

The Riemann-Stieltjes integral of a real-valued function f of a real variable with respect to a real monotone non-decreasing function g is denoted by

$$\int_a^b f(x)\, dg(x)$$

and defined to be the limit, as the mesh of the partition

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

of the interval $[a, b]$ approaches zero, of the approximating RiemannStieltjes sum

$$S(P, f, g) = \sum_{i=0}^{n-1} f(c_i)(g(x_{i+1}) - g(x_i))$$

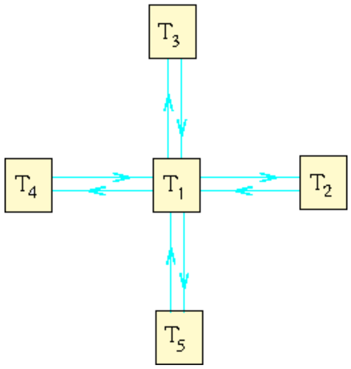
where c_i is in the i-th subinterval $[x_i, x_{i+1}]$. The two functions f and g are respectively called the **integrand** and the **integrator**.

The **limit** is a number A, the value of the Riemann-Stieltjes integral. The meaning of the limit: Given $\varepsilon > 0$, then there exists $\delta > 0$ such that for every partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ with $\mathbf{mesh}(P) = \max_{0 \leq i < n} (x_{i+1} - x_i) < \delta$, and for every choice of points c_i in $[x_i, x_{i+1}]$,

$$|S(P, f, g) - A| < \varepsilon.$$

Quiz 12

Problem 1.



Flow through each pipe is f gallons per unit time.

Each tank has constant volume V.

Symbols $x_1(t)$ to $x_5(t)$ are the salt amounts in tanks T_1 to T_5, respectively.

The differential equations are obtained by the classical balance law, which says that the rate of change in salt amount is the rate in minus the rate out. Individual rates in/out are of the form (flow rate)(salt concentration), where flow rate f has units volume per unit time and $x_i(t)/V$ is the concentration = amount/volume.

$$\begin{aligned}
x_1'(t) &= \tfrac{f}{V}\left(x_2(t) + x_3(t) + x_4(t) + x_5(t) - 4x_1(t)\right) \\
x_2'(t) &= \tfrac{f}{V}\left(x_1(t) - x_2(t)\right), \\
x_3'(t) &= \tfrac{f}{V}\left(x_1(t) - x_3(t)\right), \\
x_4'(t) &= \tfrac{f}{V}\left(x_1(t) - x_4(t)\right), \\
x_5'(t) &= \tfrac{f}{V}\left(x_1(t) - x_5(t)\right).
\end{aligned}$$

Problem 1(a). Change variables $t = Vr/f$ to obtain the new system

$$\begin{aligned}
\frac{dx_1}{dr} &= x_2 + x_3 + x_4 + x_5 - 4x_1 \\
\frac{dx_2}{dr} &= x_1 - x_2, \\
\frac{dx_3}{dr} &= x_1 - x_3, \\
\frac{dx_4}{dr} &= x_1 - x_4, \\
\frac{dx_5}{dr} &= x_1 - x_5.
\end{aligned}$$

Problem 1(b). Formulate the equations in 1(a) in the system form $\frac{d}{dr}\vec{u} = A\vec{u}$.

Answer: 1

$$A = \begin{pmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

Problem 1(c). Find the eigenvalues of A.

Answer: $\lambda = 0, -1, -1, -1, -5$

Problem 1(d). Find the eigenvectors of A.

Problem 1(e). Solve the differential equation $\frac{d\vec{u}}{dr} = A\vec{u}$ by the eigenanalysis method.

Three Methods for Solving $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$

- **Eigenanalysis Method**. The eigenpairs of matrix A are required. The matrix A must be diagonalizable, meaning there are n eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), \ldots, (\lambda_n, \vec{v}_n)$. The main theorem says that the general solution of $\vec{u}' = A\vec{u}$ is

$$\vec{u}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots + c_n e^{\lambda_n t}\vec{v}_n.$$

- **Laplace's Method**. Solve the scalar equations by the Laplace transform method. The **resolvent method** automates this process: $\vec{u}(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)\vec{u}(0)$.

- **Cayley-Hamilton-Ziebur Method**. The solution $\vec{u}(t)$ is a vector linear combination of the Euler solution atoms $f_1, \ldots, f_n$ found from the roots of the characteristic equation $|A - \lambda I| = 0$. The vectors $\vec{d}_1, \ldots, \vec{d}_n$ in the linear combination

$$\vec{u}(t) = f_1(t)\vec{d}_1 + f_2(t)\vec{u}_2 + \cdots + f_n(t)\vec{d}_n$$

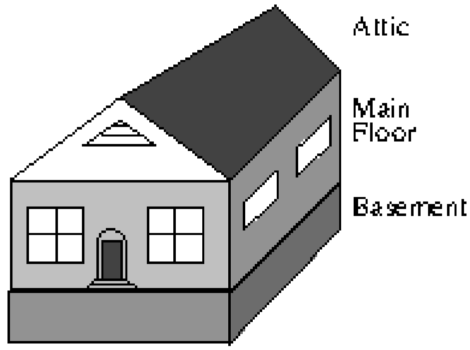
are determined by the explicit formula

$$< \vec{d}_1 \,|\, \vec{d}_2 \,|\, \cdots \,|\, \vec{d}_n > = < \vec{u}_0 \,|\, A\vec{u}_0 \,|\, \cdots \,|\, A^{n-1}\vec{u}_0 > \left(W(0)^T\right)^{-1},$$

where $W(t)$ is the Wronskian matrix of atoms $f_1, \ldots, f_n$ and $\vec{u}_0$ is the initial data.

Problem 2. Home Heating

Consider a typical home with attic, basement and insulated main floor.



Heating Assumptions and Variables

- It is usual to surround the main living area with insulation, but the attic area has walls and ceiling without insulation.
- The walls and floor in the basement are insulated by earth.
- The basement ceiling is insulated by air space in the joists, a layer of flooring on the main floor and a layer of drywall in the basement.

The changing temperatures in the three levels is modeled by Newton's cooling law and the variables

$$\begin{aligned} z(t) &= \text{Temperature in the attic,} \\ y(t) &= \text{Temperature in the main living area,} \\ x(t) &= \text{Temperature in the basement,} \\ t &= \text{Time in hours.} \end{aligned}$$

A typical mathematical model is the set of equations

$$\begin{aligned} x' &= \frac{3}{4}(45 - x) + \frac{1}{4}(y - x), \\ y' &= \frac{1}{4}(x - y) + \frac{1}{4}(40 - y) + \frac{1}{2}(z - y) + 20, \\ z' &= \frac{1}{2}(y - z) + \frac{1}{2}(35 - z). \end{aligned}$$

Problem 2(a). Formulate the system of differential equations as a matrix system $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t) + \vec{b}$. Show details.

Answer. $\vec{u} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$, $\quad \vec{b} = \begin{pmatrix} \frac{3}{4}(45) \\ 20 + \frac{40}{4} \\ \frac{35}{2} \end{pmatrix}$, $\quad A = \begin{pmatrix} -1 & \frac{1}{4} & 0 \\ \frac{1}{4} & -1 & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 \end{pmatrix}$

Problem 2(b). The heating problem has an **equilibrium solution** $\vec{u}_p(t)$ which is a constant vector of temperatures for the three floors. It is formally found by setting $\frac{d}{dt}\vec{u}(t) = 0$, and then $\vec{u}_p = -A^{-1}\vec{b}$. Justify the algebra and explicitly find $\vec{u}_p(t)$.

Answer 2(b). $\vec{u}_p(t) = -A^{-1}\vec{b} = \begin{pmatrix} \frac{560}{11} \\ \frac{755}{11} \\ \frac{570}{11} \end{pmatrix} = \begin{pmatrix} 50.91 \\ 68.64 \\ 51.82 \end{pmatrix}$.

Problem 2(c). The homogeneous problem is $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$. It can be solved by a variety of methods, three major methods enumerated below. Choose a method and solve for $\vec{x}(t)$.

Answer 2(c): The homogenous scalar general solution is

$$\begin{aligned} x_1(t) &= -2c_1e^{-t} + \frac{1}{2}c_2e^{-at} + \frac{1}{2}c_3e^{-bt}, \\ x_2(t) &= -\frac{1}{2}\sqrt{5}c_2e^{-at} - \frac{1}{2}\sqrt{5}c_3e^{-bt}, \\ x_3(t) &= c_1e^{-t} + c_2e^{-at} + c_3e^{-bt}. \end{aligned}$$

Four Methods for Solving $\vec{u}' = A\vec{u}$

- **Eigenanalysis Method**. Three eigenpairs of matrix A are required. The matrix A must be diagonalizable, meaning there are 3 eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), (\lambda_3, \vec{v}_3)$. The main theorem says that the general solution of $\vec{u}' = A\vec{u}$ is

$$\vec{u}(t) = c_1e^{\lambda_1 t}\vec{v}_1 + c_2e^{\lambda_2 t}\vec{v}_2 + c_3e^{\lambda_3 t}\vec{v}_3.$$

- **Laplace's Method**. Solve the scalar equations by the Laplace transform method. The **resolvent method** automates this process: $\vec{u}(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)\vec{u}(0)$.

- **Cayley-Hamilton-Ziebur Method**. The solution $\vec{u}(t)$ is a vector linear combination

$$\vec{u}(t) = \vec{d}_1 f_1(t) + \vec{d}_2 f_2(t) + \vec{d}_3 f_3(t)$$

  of the Euler solution atoms f_1, f_2, f_3 found from the roots of the characteristic equation $|A - \lambda I| = 0$.

  The vectors $\vec{d}_1, \vec{d}_2, \vec{d}_3$ are determined by the explicit formula

$$< \vec{d}_1 \,|\, \vec{d}_2 \,|\, \vec{d}_3 > = < \vec{u}_0 \,|\, A\vec{u}_0 \,|\, A^2\vec{u}_0 > \left(W(0)^T\right)^{-1},$$

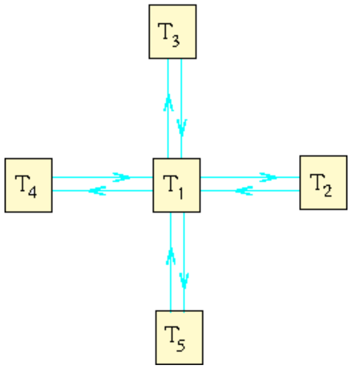
  where $W(t)$ is the Wronskian matrix of atoms f_1, f_2, f_3 and $\vec{u}_0$ is the initial data.

- **Exponential Matrix Method**. The method uses e^{At}, which is a fundamental matrix $\Phi(t)$ for the system $\frac{d}{dt}\vec{u} = A\vec{u}$, which satisfies the extra condition $\Phi(0) = I = n \times n$ identity matrix. Then the solution of the system can be written $\vec{u}(t) = e^{At}\vec{u}(0)$. Putzer's method applies to find e^{At} in any dimension n. More practical is a computer algebra system. For instance, `maple` finds the exponential matrix by this sample code:

```
A:=<1,2|3,4>; LinearAlgebra[MatrixExponential](A,t);
```

Quiz 13

Problem 1.



Flow through each pipe is f gallons per unit time.

Each tank has constant volume V.

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The differential equations are obtained by the classical balance law, which says that the rate of change in salt amount is the rate in minus the rate out. Individual rates in/out are of the form (flow rate)(salt concentration), where flow rate f has units volume per unit time and $x_i(t)/V$ is the concentration = amount/volume.

$$\begin{aligned}
x_1'(t) &= \frac{f}{V}\left(x_2(t) + x_3(t) + x_4(t) + x_5(t) - 4x_1(t)\right) \\
x_2'(t) &= \frac{f}{V}\left(x_1(t) - x_2(t)\right), \\
x_3'(t) &= \frac{f}{V}\left(x_1(t) - x_3(t)\right), \\
x_4'(t) &= \frac{f}{V}\left(x_1(t) - x_4(t)\right), \\
x_5'(t) &= \frac{f}{V}\left(x_1(t) - x_5(t)\right).
\end{aligned}$$

Problem 1(a). Change variables $t = Vr/f$ to obtain the new system

$$\begin{aligned}
\frac{dx_1}{dr} &= x_2 + x_3 + x_4 + x_5 - 4x_1 \\
\frac{dx_2}{dr} &= x_1 - x_2, \\
\frac{dx_3}{dr} &= x_1 - x_3, \\
\frac{dx_4}{dr} &= x_1 - x_4, \\
\frac{dx_5}{dr} &= x_1 - x_5.
\end{aligned}$$

Problem 1(b). Formulate the equations in 1(a) in the system form $\frac{d}{dr}\vec{u} = A\vec{u}$.

Answer:

$$A = \begin{pmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

Problem 1(c). Find the eigenvalues of A.

Answer: $\lambda = 0, -1, -1, -1, -5$

Problem 1(d). Find the eigenvectors of A.

Problem 1(e). Solve the differential equation $\frac{d\vec{u}}{dr} = A\vec{u}$ by the eigenanalysis method.

Three Methods for Solving $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$

- **Eigenanalysis Method**. The eigenpairs of matrix A are required. The matrix A must be diagonalizable, meaning there are n eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), \ldots, (\lambda_n, \vec{v}_n)$. The main theorem says that the general solution of $\vec{u}' = A\vec{u}$ is

$$\vec{u}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots + c_n e^{\lambda_n t}\vec{v}_n.$$

- **Laplace's Method**. Solve the scalar equations by the Laplace transform method. The **resolvent method** automates this process: $\vec{u}(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)\vec{u}(0)$.

- **Cayley-Hamilton-Ziebur Method**. The solution $\vec{u}(t)$ is a vector linear combination of the Euler solution atoms $f_1, \ldots, f_n$ found from the roots of the characteristic equation $|A - \lambda I| = 0$. The vectors $\vec{d}_1, \ldots, \vec{d}_n$ in the linear combination

$$\vec{u}(t) = f_1(t)\vec{d}_1 + f_2(t)\vec{u}_2 + \cdots + f_n(t)\vec{d}_n$$

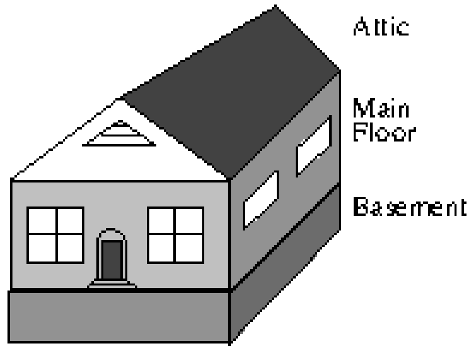
are determined by the explicit formula

$$< \vec{d}_1 \,|\, \vec{d}_2 \,|\, \cdots \,|\, \vec{d}_n > = < \vec{u}_0 \,|\, A\vec{u}_0 \,|\, \cdots \,|\, A^{n-1}\vec{u}_0 > \left(W(0)^T\right)^{-1},$$

where $W(t)$ is the Wronskian matrix of atoms $f_1, \ldots, f_n$ and $\vec{u}_0$ is the initial data.

Problem 2. Home Heating

Consider a typical home with attic, basement and insulated main floor.



Heating Assumptions and Variables

- It is usual to surround the main living area with insulation, but the attic area has walls and ceiling without insulation.
- The walls and floor in the basement are insulated by earth.
- The basement ceiling is insulated by air space in the joists, a layer of flooring on the main floor and a layer of drywall in the basement.

The changing temperatures in the three levels is modeled by Newton's cooling law and the variables

$$\begin{aligned} z(t) &= \text{Temperature in the attic,} \\ y(t) &= \text{Temperature in the main living area,} \\ x(t) &= \text{Temperature in the basement,} \\ t &= \text{Time in hours.} \end{aligned}$$

A typical mathematical model is the set of equations

$$\begin{aligned} x' &= \frac{3}{4}(45 - x) + \frac{1}{4}(y - x), \\ y' &= \frac{1}{4}(x - y) + \frac{1}{4}(40 - y) + \frac{1}{2}(z - y) + 20, \\ z' &= \frac{1}{2}(y - z) + \frac{1}{2}(35 - z). \end{aligned}$$

Problem 2(a). Formulate the system of differential equations as a matrix system $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t) + \vec{b}$. Show details.

Answer. $\vec{u} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} \frac{3}{4}(45) \\ 20 + \frac{40}{4} \\ \frac{35}{2} \end{pmatrix}, \quad A = \begin{pmatrix} -1 & \frac{1}{4} & 0 \\ \frac{1}{4} & -1 & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 \end{pmatrix}$

Problem 2(b). The heating problem has an **equilibrium solution** $\vec{u}_p(t)$ which is a constant vector of temperatures for the three floors. It is formally found by setting $\frac{d}{dt}\vec{u}(t) = 0$, and then $\vec{u}_p = -A^{-1}\vec{b}$. Justify the algebra and explicitly find $\vec{u}_p(t)$.

Answer 2(b). $\vec{u}_p(t) = -A^{-1}\vec{b} = \begin{pmatrix} \frac{560}{11} \\ \frac{755}{11} \\ \frac{570}{11} \end{pmatrix} = \begin{pmatrix} 50.91 \\ 68.64 \\ 51.82 \end{pmatrix}.$

Problem 2(c). The homogeneous problem is $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$. It can be solved by a variety of methods, three major methods enumerated below. Choose a method and solve for $\vec{x}(t)$.

Answer 2(c): The homogenous scalar general solution is

$$\begin{aligned} x_1(t) &= -\frac{1}{2}c_1e^{-t} + 2c_2e^{-at} + 2c_3e^{-bt}, \\ x_2(t) &= -\sqrt{5}c_2e^{-at} + \sqrt{5}c_3e^{-bt}, \\ x_3(t) &= c_1e^{-t} + c_2e^{-at} + c_3e^{-bt}. \end{aligned}$$

Three Methods for Solving $\vec{u}' = A\vec{u}$

- **Eigenanalysis Method**. Three eigenpairs of matrix A are required. The matrix A must be diagonalizable, meaning there are 3 eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), (\lambda_3, \vec{v}_3)$. The main theorem says that the general solution of $\vec{u}' = A\vec{u}$ is

$$\vec{u}(t) = c_1e^{\lambda_1 t}\vec{v}_1 + c_2e^{\lambda_2 t}\vec{v}_2 + c_3e^{\lambda_3 t}\vec{v}_3.$$

- **Laplace's Method**. Solve the scalar equations by the Laplace transform method. The **resolvent method** automates this process: $\vec{u}(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)\vec{u}(0)$.

- **Cayley-Hamilton-Ziebur Method**. The solution $\vec{u}(t)$ is a vector linear combination

$$\vec{u}(t) = \vec{d}_1 f_1(t) + \vec{d}_2 f_2(t) + \vec{d}_3 f_3(t)$$

  of the Euler solution atoms f_1, f_2, f_3 found from the roots of the characteristic equation $|A - \lambda I| = 0$.

  The vectors $\vec{d}_1, \vec{d}_2, \vec{d}_3$ are determined by the explicit formula

$$< \vec{d}_1 \,|\, \vec{d}_2 \,|\, \vec{d}_3 >=< \vec{u}_0 \,|\, A\vec{u}_0 \,|\, A^2\vec{u}_0 > \left(W(0)^T\right)^{-1},$$

  where $W(t)$ is the Wronskian matrix of atoms f_1, f_2, f_3 and $\vec{u}_0$ is the initial data.