Variation of Parameters

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Variation of Parameters

The **method of variation of parameters** applies to solve

$$a(x)y'' + b(x)y' + c(x)y = f(x). \tag{1}$$

- Continuity of a, b, c and f is assumed, plus $a(x) \neq 0$.
- This method solves the largest class of equations.
- Specifically *included* are functions $f(x)$ like $\ln|x|$, $|x|$, e^{x^2}.
- The method of undetermined coefficients can only succeed for $f(x)$ equal to a linear combination of atoms. It fails for functions $\ln|x|$, $|x|$, e^{x^2}.
- Variation of parameters succeeds for all the cases skipped by the method of undetermined coefficients.

Homogeneous Equation

The method of variation of parameters uses facts about the homogeneous differential equation

(2)
$$a(x)y'' + b(x)y' + c(x)y = 0.$$

Success in the method depends upon writing the general solution of (2) as

(3)
$$y = c_1 y_1(x) + c_2 y_2(x)$$

where y_1, y_2 are *known functions* and c_1, c_2 are arbitrary constants.

If a, b, c are constants, then Euler's Theorems implies y_1 and y_2 are *independent atoms*.

Typical answers for second order equations

$y = c_1 e^x + c_2 e^{-x}$ (distinct roots $r = 1$, $r = -1$.)

$y = c_1 e^x + c_2 x e^x$ (double root $r = 1$)

$y = c_1 + c_2 x$ (double root $r = 0$)

$y = c_1 e^x \cos 2x + c_2 e^x \sin 2x$ (complex roots $1 \pm 2i$)

$y = c_1 \cos 2x + c_2 \sin 2x$ (complex roots $0 \pm 2i$)

Independence

Two solutions y_1, y_2 of $a(x)y'' + b(x)y' + c(x)y = 0$ are called **independent** if neither is a constant multiple of the other. The term **dependent** means *not independent*, in which case either $y_1(x) = cy_2(x)$ or $y_2(x) = cy_1(x)$ holds for all x, for some constant c.

Independence can be tested through the **Wronskian** of y_1, y_2, defined by

$$W(x) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

Linear algebra supplies one result:

Theorem 1 (Wronskian Test for Independence)
Assume the Wronskian of two solutions $y_1(x)$, $y_2(x)$ is nonzero at some $x = x_0$. Then $y_1(x)$, $y_2(x)$ are independent.

Abel's Identity and the Wronskian Test

Theorem 2 (Wronskian and Independence)
The Wronskian of two solutions satisfies the homogeneous first order differential equation

$$a(x)W' + b(x)W = 0.$$

This implies **Abel's identity**

$$W(x) = \frac{W(x_0)}{e^{\int_{x_0}^{x}(b(t)/a(t))dt}}.$$

Theorem 3 (Second Order DE Wronskian Test)
Two solutions of $a(x)y'' + b(x)y' + c(x)y = 0$ are independent if and only if their Wronskian is nonzero at some point x_0.

Variation of Parameters Formula

Theorem 4 (Variation of Parameters Formula)
Let a, b, c, f be continuous near $x = x_0$ and $a(x) \neq 0$. Let y_1, y_2 be two independent solutions of the homogeneous equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

with computed Wronskian $W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$. Then a particular solution $y_p(x)$ of the non-homogeneous differential equation

$$a(x)y'' + b(x)y' + c(x)y = f(x)$$

can be computed by substituting the four expressions y_1, y_2, W and f into the formula

$$y_p(x) = \left(\int \frac{y_2(x)(-f(x))}{a(x)W(x)}dx\right) y_1(x) + \left(\int \frac{y_1(x)f(x)}{a(x)W(x)}dx\right) y_2(x).$$

The variation of parameters formula is so named because it expresses $y_p = c_1y_1 + c_2y_2$, where c_1 and c_2 are functions of x, whereas $y_h = c_1y_1 + c_2y_2$ with c_1, c_2 constants.

1 Example (Independence) Consider $y'' - y = 0$. Show the two solutions $\sinh(x)$ and $\cosh(x)$ are independent using Wronskians.

Solution. Let $W(x)$ be the Wronskian of $\sinh(x)$ and $\cosh(x)$. The calculation below shows $W(x) = -1$. By Theorem 2, the solutions are independent.
Background. The calculus *definitions* for hyperbolic functions are

$$\sinh x = (e^x - e^{-x})/2, \quad \cosh x = (e^x + e^{-x})/2.$$

Their derivatives are $(\sinh x)' = \cosh x$ and $(\cosh x)' = \sinh x$. For instance, $(\cosh x)'$ stands for $\frac{1}{2}(e^x + e^{-x})'$, which evaluates to $\frac{1}{2}(e^x - e^{-x})$, or $\sinh x$.

Wronskian detail.
Let $y_1 = \sinh x$, $y_2 = \cosh x$. Then

$W = y_1(x)y_2'(x) - y_1'(x)y_2(x)$	Definition of Wronskian W.
$= \sinh(x)\sinh(x) - \cosh(x)\cosh(x)$	Substitute for y_1, y_1', y_2, y_2'.
$= \frac{1}{4}(e^x - e^{-x})^2 - \frac{1}{4}(e^x + e^{-x})^2$	Apply exponential definitions.
$= -1$	Expand and cancel terms.

2 Example (Wronskian) Given $2y'' - xy' + 3y = 0$, verify that a solution pair y_1, y_2 has Wronskian $W(x) = W(0)e^{x^2/4}$.

Solution

Let $a(x) = 2$, $b(x) = -x$, $c(x) = 3$. The Wronskian is a solution of

$$W' = -(b/a)W.$$

Then $W' = xW/2$. The solution is a constant divided by the integrating factor $e^{\int -(x/2)dx}$. Resolving the constant from the initial condition for $W(x)$ implies

$$W = W(0)e^{x^2/4}.$$

3 Example (Variation of Parameters) Solve $y'' + y = \sec x$ by variation of parameters, verifying $y = c_1 \cos x + c_2 \sin x + x \sin x + \cos(x) \ln|\cos x|$.

Solution

Homogeneous solution y_h. Euler's method is applied for constant equation $y'' + y = 0$. The characteristic equation $r^2 + 1 = 0$ has roots $r = \pm i$, hence the atoms are $\cos x$, $\sin x$. Then $y_h(x) = c_1 \cos x + c_2 \sin x$.

Wronskian. Suitable independent solutions are $y_1 = \cos x$ and $y_2 = \sin x$, taken from the general solution of the homogeneous equation $y_h(x) = c_1 \cos x + c_2 \sin x$. Then $W(x) = \cos^2 x + \sin^2 x = 1$.

Calculate y_p. The variation of parameters formula (4) applies. Integration proceeds near $x = 0$, because $\sec(x)$ is continuous near $x = 0$.

$$\begin{aligned} y_p(x) &= -y_1(x) \int y_2(x) \sec(x) dx + y_2(x) \int y_1(x) \sec x dx & \boxed{1} \\ &= -\cos x \int \tan(x) dx + \sin x \int 1 dx & \boxed{2} \\ &= x \sin x + \cos(x) \ln|\cos x| & \boxed{3} \end{aligned}$$

Details: $\boxed{1}$ Use equation (4). $\boxed{2}$ Substitute $y_1 = \cos x$, $y_2 = \sin x$. $\boxed{3}$ Integral tables applied. Integration constants set to zero.