Applications of Systems of Differential Equations

- **Brine Tank Cascade**
- **Cascade Model**
- **Recycled Brine Tank Cascade**
- **Recycled Cascade Model**

Brine Tank Cascade

Let brine tanks A, B, C be given of volumes 20, 40, 60, respectively, as in Figure 1.

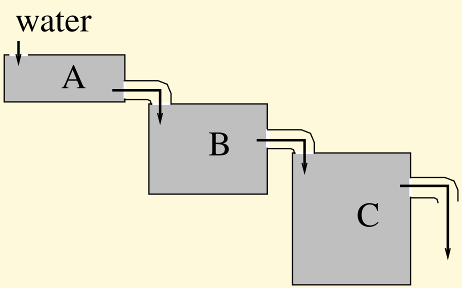

Figure 1. Three brine tanks in cascade.

Assumptions and Notation

- It is supposed that fluid enters tank A at rate r, drains from A to B at rate r, drains from B to C at rate r, then drains from tank C at rate r. Hence the volumes of the tanks remain constant. Let $r = 10$, to illustrate the ideas.
- Uniform stirring of each tank is assumed, which implies **uniform salt concentration** throughout each tank.
- Let $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the amount of salt at time t in each tank. We suppose added to tank A **water containing no salt**. Therefore, the salt in all the tanks is eventually lost from the drains.

Cascade Model

The cascade is modeled by the **chemical balance law**

$$\text{rate of change} \quad = \quad \text{input rate} \quad - \quad \text{output rate}.$$

Application of the balance law results in the triangular differential system

$$\begin{aligned} x_1' &= -\frac{1}{2}x_1, \\ x_2' &= \frac{1}{2}x_1 - \frac{1}{4}x_2, \\ x_3' &= \frac{1}{4}x_2 - \frac{1}{6}x_3. \end{aligned}$$

Cascade Model Solution

The solution is justified by the integrating factor method for first order scalar differential equations.

$$\begin{aligned}
x_1(t) &= x_1(0)e^{-t/2}, \\
x_2(t) &= -2x_1(0)e^{-t/2} + (x_2(0) + 2x_1(0))e^{-t/4}, \\
x_3(t) &= \frac{3}{2}x_1(0)e^{-t/2} - 3(x_2(0) + 2x_1(0))e^{-t/4} \\
&\quad + (x_3(0) - \frac{3}{2}x_1(0) + 3(x_2(0) + 2x_1(0)))e^{-t/6}.
\end{aligned}$$

Recycled Brine Tank Cascade

Let brine tanks A, B, C be given of volumes 60, 30, 60, respectively, as in Figure 2.

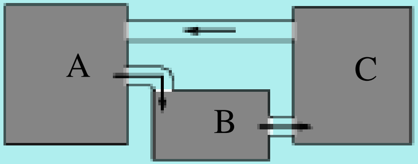

Figure 2. Three brine tanks in cascade with recycling.

Assumptions and Notation

- Suppose that fluid drains from tank A to B at rate r, drains from tank B to C at rate r, then drains from tank C to A at rate r. The tank volumes remain constant due to constant recycling of fluid. For purposes of illustration, let $r = 10$.
- Uniform stirring of each tank is assumed, which implies **uniform salt concentration** throughout each tank.
- Let $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the amount of salt at time t in each tank. No salt is lost from the system, due to recycling.

Recycled Cascade Model

Using compartment analysis, the recycled cascade is modeled by the non-triangular system

$$\begin{aligned} x_1' &= -\frac{1}{6}x_1 && && + \frac{1}{6}x_3, \\ x_2' &= \frac{1}{6}x_1 && - \frac{1}{3}x_2, \\ x_3' &= && \frac{1}{3}x_2 && - \frac{1}{6}x_3. \end{aligned}$$

Recycled Cascade Solution

$$
\begin{aligned}
x_1(t) &= c_1 + (c_2 - 2c_3)e^{-t/3}\cos(t/6) + (2c_2 + c_3)e^{-t/3}\sin(t/6), \\
x_2(t) &= \frac{1}{2}c_1 + (-2c_2 - c_3)e^{-t/3}\cos(t/6) + (c_2 - 2c_3)e^{-t/3}\sin(t/6), \\
x_3(t) &= c_1 + (c_2 + 3c_3)e^{-t/3}\cos(t/6) + (-3c_2 + c_3)e^{-t/3}\sin(t/6).
\end{aligned}
$$

- At infinity, $x_1 = x_3 = c_1$, $x_2 = c_1/2$. The meaning is that the total amount of salt is uniformly distributed in the tanks, in the ratio $2:1:2$.
- The solution of the system was found by the eigenanalysis method. It can also be found by the resolvent method in Laplace theory or the Cayley-Hamilton-Ziebur method.